Three-Reggeon ladders and four-loop amplitudes in the high-energy limit

HEP Remote Seminar – IIT Hyderabad

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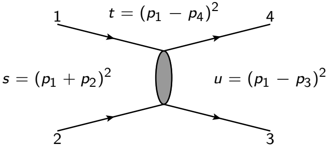
Amplitudes in the high-energy limit



High-energy limit

$$s \gg -t$$
$$u \simeq -s$$

Large logarithms in the amplitude

$$L = \underbrace{\frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right)}_{\textbf{Signature}\ \text{even}}$$

Expansion in loops ($a_s = \alpha_s/\pi$) and **towers** of logarithms

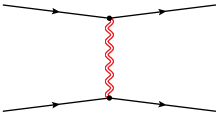
$$\mathcal{M} = \mathcal{M}^0 + a_s L\, \mathcal{M}^{(1,1)} + a_s\, \mathcal{M}^{(1,0)}$$
$$+\, a_s^2 L^2\, \mathcal{M}^{(2,2)} + a_s^2 L\, \mathcal{M}^{(2,1)} + a_s^2\, \mathcal{M}^{(2,0)}$$

LL	NLL	NNLL
$a_s L\, \mathcal{M}^{(1,1)}$	$a_s\, \mathcal{M}^{(1,0)}$	
$a_s^2 L^2\, \mathcal{M}^{(2,2)}$	$a_s^2 L\, \mathcal{M}^{(2,1)}$	$a_s^2\, \mathcal{M}^{(2,0)}$

Gluon reggeization

Leading tower: one-Reggeon exchange in the t-channel
(Lipatov;Fadin,Kuraev,Lipatov 1976)

$$\frac{1}{t} \to \frac{1}{t}\left(\frac{s}{-t}\right)^{\frac{a_s C_A}{\epsilon} r_\Gamma}$$



Leading Logarithms of the amplitude

$$\mathcal{M}^{\rm LL} = \mathcal{M}^0\, e^{\frac{a_s C_A L}{\epsilon} r_\Gamma} \qquad r_\Gamma = e^{\epsilon\gamma_E}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$$

- **All orders** in the coupling constant
- **Universality**: dependence on the process via $\mathcal{M}^0$

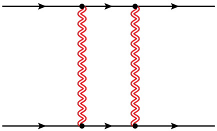
Beyond the leading tower

Amplitudes have **real** and **imaginary** parts beyond LL

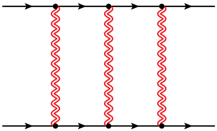
Deep connection between analicity and signature

- $\mathcal{M}^{(+)}$ is the **imaginary** part and is **even**
- $\mathcal{M}^{(-)}$ is the **real** part and is **odd**

Signature depends on how many Reggeons we exchange



$\mathcal{M}^{(+)}$ **even** exchange



$\mathcal{M}^{(-)}$ **odd** exchange.

I will focus on the **odd** amplitudes.

Real (odd) amplitudes

- High-energy limit
- Amplitudes from Evolution
- Computing four-loop amplitudes
- Results and outlook

At NLL $\mathcal{M}^{(-)}$ is still given by a single-Reggeon exchange (Fadin, Kozlov, Reznichenko 2015).

Three-Reggeons states appear at NNLL

Observed in two-loop amplitudes (Del Duca, Glover 2001).
Recent investigations up to three loops

- IR singularities (Del Duca, G.F, Magnea, Vernazza 2013-14; Fadin 2016; Fadin, Lipatov 2017)
- Complete result (Caron-Huot, Gardi, Vernazza 2017)

Four-loop amplitudes

The Regge limit opens a window on **four loop** amplitudes

$$\mathcal{M} = \mathbf{Z}\,\mathcal{H},$$

- $\mathcal{H}$ **finite** and **process-dependent**
- **Z IR-divergent** and **universal**

$$\mathbf{Z} = \mathcal{P}\exp\left[-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\boldsymbol{\Gamma}\right]$$

$\boldsymbol{\Gamma}$ fundamental quantity in gauge theory, currently known to three loops (Almelid,Duhr,Gardi 2015)

The four-loop soft anomalous dimension

Recently proposed **most general** ansatz for $\boldsymbol{\Gamma}$ at 4 loops (Becher,Neubert 2019)

$$\boldsymbol{\Gamma}^{(4)} = \sum_R \Big\{ g^R \Big[\sum_{(i,j)} \Big(\mathcal{D}^R_{iijj} + 2\mathcal{D}^R_{iiij} \Big) \log \frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)} \mathcal{D}^R_{ijkk} \log \frac{\mu^2}{-s_{ij}} \Big] + \mathcal{D}^R_{ijkl} G^R \Big\} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkli} H_1 + \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm} H_2 .$$

g^R, G^R, H_1 and H_2 arbitrary functions.

Can we probe these structures in the high-energy limit?
Let's compute the four-loop amplitudes!

Our questions

Infrared singularities

Compute $\mathbf{\Gamma}^{(4)}$ in the high-energy limit.

- Constrain the ansatz (Becher,Neubert 2019).
- Regge limit key to bootstrap $\mathbf{\Gamma}^{(3)}$ (Almelid,Duhr,Gardi,McLeod,White 2018)

Finite parts

In the Regge limit finite parts are **universal** too.

- Get the complete amplitudes, including $\mathcal{H}$

Structure of the Regge limit

The NNLL tower shows triple-Reggeon exchange

- New colour structure, beyond $\mathcal{M}^0$
- Resummation of the NNLL tower

Wilson lines in high-energy scattering

Fast-moving particles → Wilson lines close to lightcone

(Korchemsky, Korchemskaya 1995; Balitsky 1996)

$$U^{\eta}(z) = \mathcal{P} \exp\left[i g_s \mathbf{T}^a \int_{-\infty}^{+\infty} dx^+ \, A_+^a(x^+, x^- = 0, z) \right]$$

- $\mathbf{T}^a$ group generator in the parton representation.
- $\eta = L$ (implicit) rapidity cutoff.

$$\frac{d}{dL} U(z_1) \dots U(z_n) = \sum_{i,j=1}^{n} H_{ij} \, U(z_1) \dots U(z_n)$$

- H_{ij} Balitsky-JIMWLK hamiltonian.

High-energy logarithms L **predicted** by evolution!

The Reggeon field

- High-energy limit
- Amplitudes from Evolution
- Computing four-loop amplitudes
- Results and outlook

Reggeon field defined in terms of U^η (Caron-Huot 2013)

$$U^\eta(z) = 1 + ig_s \mathbf{T}^a W^a(z) - \frac{g_s^2}{2}\mathbf{T}^a\mathbf{T}^b W^a(z) W^b(z) + \dots$$
$$= \exp\left[ig_s \mathbf{T}^a W^a\right]$$

- Reggeons are **signature-odd**, $W^a \to -W^a$
  - Emission of even n. of Reggeons $\to \mathcal{M}^{(+)}$
  - Emission of odd n. of Reggeons $\to \mathcal{M}^{(-)}$

Reggeons W^η obey evolution equation in rapidity, following from the Balitsky-JIMWLK evolution of U^η.

Reggeon evolution

It is useful to think about a generic multi-Reggeon state and *translate* the Balitsky-JIMWLK hamiltonian (Caron-Huot 2013; Caron-Huot,Gardi,Vernazza 2017)

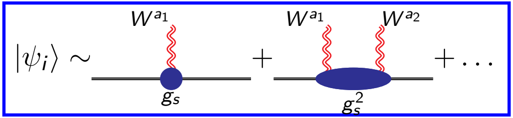
$$|\psi\rangle = \begin{pmatrix} W^{a_1} \\ W^{a_1} W^{a_2} \\ W^{a_1} W^{a_2} W^{a_3} \end{pmatrix}, \qquad \frac{d}{dL}|\psi\rangle = -H\,|\psi\rangle$$

To leading-order in a_s

$$H = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \dots \\ 0 & H_{2\to2} & 0 & \dots \\ H_{1\to3} & 0 & H_{3\to3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

- Transition **even-odd** are **forbidden**
- Diagonal $\mathcal{O}(a_s)$ vs Off-diagonal elements $\mathcal{O}(a_s^2)$

Expansion of the scattering states



The blobs are generalised couplings, or **impact factors**

$$|\psi\rangle = \underbrace{ig_s\, D_1(p)\mathbf{W}(p)}_{|\psi_1\rangle} - \underbrace{\frac{g_s^2}{2}\int_q D_2(q,p)\,\mathbf{W}(q)\mathbf{W}(p-q)}_{-|\psi_2\rangle}$$

with $\mathbf{W}(p) = \mathbf{T}^a W^a(p)$ and

$$\langle W^a(q)|W^b(p)\rangle = \frac{i}{p^2}\delta^{ab}\underbrace{\delta^{2-2\epsilon}(p-q)}_{\text{transverse mom.}} + \mathcal{O}(a_s)$$

The reduced amplitude

- High-energy limit
- Amplitudes from Evolution
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Amplitudes are target-projectile contractions

Reduced amplitude

$$\frac{i}{2s}\hat{\mathcal{M}} = \langle\psi_j|e^{-L\hat{H}}|\psi_i\rangle,$$

Tree level normalisation $i/(2s)$. **Reduced hamiltonian** (Caron-Huot, Gardi, Vernazza 2017)

$$\hat{H} = H - H_{1\to1}, \qquad \hat{H}_{1\to1} \stackrel{\text{def}}{=} 0$$

free of **single Reggeon** evolution. Complete amplitude

$$\frac{i}{2s}\mathcal{M} = \underbrace{Z_i Z_j}_{\text{Collinear}}\, e^{-LH_{1\to1}}\,\hat{\mathcal{M}}$$

Power counting rules

- High-energy limit
- Amplitudes from Evolution
- Computing four-loop amplitudes
- Results and outlook

$\hat{\mathcal{M}}^{(-)}$ involves $|\psi_1\rangle$, $|\psi_3\rangle \ldots$ and their transitions.

Rule 1: $|\psi_1\rangle \sim \mathcal{O}\left(g_s\right)$ **vs** $|\psi_3\rangle \sim \mathcal{O}\left(g_s\, a_s\right)$

$$\frac{i}{2s}\hat{\mathcal{M}}^{2-\text{loop}} = \langle\psi_{j,3}|\psi_{i,3}\rangle^{(\text{LO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{NNLO})}$$

Rule 2: $\hat{H}_{3\to3} \to \mathcal{O}(a_s L)$ **vs** $\hat{H}_{1\to3} \to \mathcal{O}(a_s^2 L)$

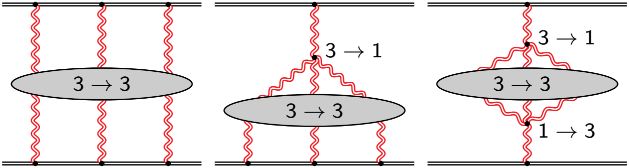
$$\frac{i}{2s}\hat{\mathcal{M}}^{3-\text{loop}} = -L\Big[\langle\psi_{j,3}|\hat{H}_{3\to3}|\psi_{i,3}\rangle + \langle\psi_{j,1}|\hat{H}_{3\to1}|\psi_{i,3}\rangle + \langle\psi_{j,3}|\hat{H}_{1\to3}|\psi_{i,1}\rangle\Big]^{(\text{LO})} + \langle\psi_{j,3}|\psi_{i,3}\rangle^{(\text{NLO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{N}^3\text{LO})}$$

The NNLL reduced amplitude

If we **restrict** to NNLL in the amplitudes

- $\hat{H}_{3\to1}$ or $\hat{H}_{1\to3}$ can be applied **at most twice**
- The emission of more than 3 Reggeons is **forbidden**

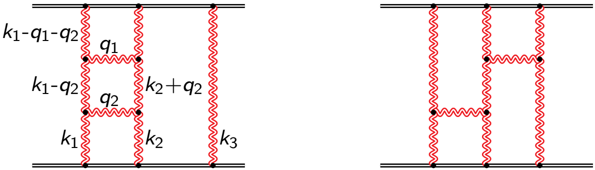
To all loop orders, $\hat{\mathcal{M}}^{(-,\mathrm{NNLL})}$ is the sum of



$\langle j_3|\left(\hat{H}_{3\to3}\right)^k|i_3\rangle$ $\langle j_1|\hat{H}_{3\to1}\left(\hat{H}_{3\to3}\right)^k|i_3\rangle$ $\langle j_1|\hat{H}_{3\to1}\left(\hat{H}_{3\to3}\right)^k\hat{H}_{1\to3}|i_1\rangle$

Three-Reggeon ladders

Two independent contributions at four loops



- Single kinematic scale $t = -p^2$.

All the integrals are massless 4-loop propagators: ✓
Diagrams indicate the iteration of

$$\hat{H}_{22}(q; k_1, k_2) = \frac{(k_1 + k_2)^2}{k_1^2 k_2^2} - \frac{(k_1 + q)^2}{k_1^2 q^2} - \frac{(k_2 - q)^2}{q^2 k_2^2}$$

The labeled diagram gives

$$\hat{H}_{22}\left(q_2; k_1 - q_2, k_2 + q_2\right) \hat{H}_{22}\left(q_2; k_1 - q_2 - q_1, k_2 + q_2 + q_1\right)$$

A colour puzzle

The integrals give $\epsilon-$expansions of the result e.g.

$$M_{\mathrm{DL}} = i\frac{a_s^4 r_\Gamma^4}{\pi^2}\,\frac{1}{t}\left[\frac{5}{12\epsilon^4} - \frac{265}{6\epsilon}\zeta_3 + \dots\right] \underbrace{f^{a x_1 l} f^{l y_1 b} f^{x_1 x_2 m} f^{m y_2 y_1}}_{\text{Three-gluon vertices}}$$

$$\times\, \mathbf{T}_i^{\{a}\mathbf{T}_i^{\;b}\mathbf{T}_i^{c\}}\,\mathbf{T}_j^{\{x_2}\mathbf{T}_j^{\;y_2}\mathbf{T}_j^{c\}},$$

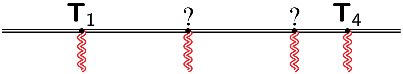
What about colour?

- Assign **specific representations** for $\mathbf{T}_i$, $\mathbf{T}_j = q,g$?
  - ✗ Obscures **universality**
  - ✗ Extraction of $\mathbf{Z}$
  - ✗ Resummation of the NNLL tower
- Write the generators of target and projectile as

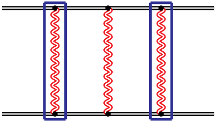
$$\mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2 \quad \mathbf{T}_t = \mathbf{T}_1 + \mathbf{T}_4 \quad \mathbf{T}_u = \mathbf{T}_1 + \mathbf{T}_3$$

Colour techniques

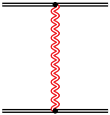
Outmost generators clearly associated to external particles



At lowest order there is no ambiguity

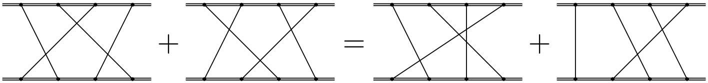


$$= \left[\frac{1}{2}\left(\mathbf{T}^2_{s-u} - \frac{\mathbf{T}^2_t}{2}\right)\right]^2 \cdot$$



where $\mathbf{T}^2_{s-u} = \frac{\mathbf{T}^2_s - \mathbf{T}^2_u}{2}$.

Reduce *entangled* configurations using identities such as



Three-Reggeon Ladders - Result

$$\langle j_3 | \hat{H}^2_{3\to 3} | i_3 \rangle = \frac{1}{144} \left[\frac{\mathbf{C}_{33}^{(4,-4)}}{\epsilon^4} + \frac{2f_\epsilon}{\epsilon} \mathbf{C}_{33}^{(4,-1)} + \mathcal{O}(\epsilon) \right] \mathcal{M}^{(0)}$$

- $f_\epsilon = \zeta_3 + \frac{3}{2}\epsilon\,\zeta_4$ (appearing in **every term** at NNLL!)
- Colour operators $\mathbf{T}_t^2$ and $\mathbf{T}_{s-u}^2$ acting on $\mathcal{M}^{(0)}$
- Contribution of quartic Casimir

Three-Reggeon Ladders - Result

- High-energy limit
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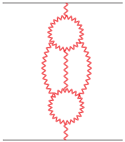
- Colour operators $\mathbf{T}_t^2$ and $\mathbf{T}_{s-u}^2$ acting on $\mathcal{M}^{(0)}$
- Contribution of quartic Casimir

$$\begin{aligned}
\mathbf{C}_{33}^{(4,-4)} &= 6\left(17C_A\mathbf{T}_t^2 - 6C_A^2 - 6(\mathbf{T}_t^2)^2\right)\mathbf{C}_{33}^{(2)} \\
&\quad - \frac{3}{4}\mathbf{T}_{s-u}^2(\mathbf{T}_t^2)^2\mathbf{T}_{s-u}^2 + \frac{25}{144}C_A^4 + \frac{1}{3}\frac{d_{AA}}{N_A} - 3C_A\left(\frac{d_{AR_i}}{N_{R_i}} + \frac{d_{AR_j}}{N_{R_j}}\right) \\
\mathbf{C}_{33}^{(4,-1)} &= 18\left(521\,C_A\mathbf{T}_t^2 - 300C_A^2 - 220(\mathbf{T}_t^2)^2\right)\mathbf{C}_{33}^{(2)} - 101\,\mathbf{C}_{33}^{(4,-4)} \\
\mathbf{C}_{33}^{(2)} &= \frac{1}{24}\left(\mathbf{T}_{s-u}^2 - \frac{C_A^2}{12}\right)
\end{aligned}$$

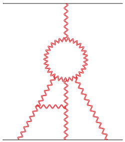
Transitions to single Reggeons

To all orders, terms with a single Reggeon are $\propto \mathcal{M}^{(0)}$

- Colour must flow through a single Reggeon



$$= \frac{1}{432}\left[-\left(\frac{C_A^4}{12} + \frac{d_{AA}}{N_A}\right)\frac{1}{\epsilon^4} + \left(\frac{101}{6}C_A^4 + 220\frac{d_{AA}}{N_A}\right)\frac{f_\epsilon}{\epsilon}\right]\mathcal{M}^{(0)}$$



$$= \frac{C_A}{144}\frac{d_{AR_i}}{N_{R_i}}\left[\frac{1}{\epsilon^4} - 208\frac{f_\epsilon}{\epsilon}\right]\mathcal{M}^{(0)}$$

Complete Reduced amplitude

- High-energy limit
- Amplitudes from Evolution
- Computing four-loop amplitudes
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A four-loop amplitude (almost) fitting in one line

$$\hat{\mathcal{M}}^{(-,4,2)} = \frac{r_\Gamma^4 \pi^2}{144}\left[\frac{\mathbf{C}_{\mathcal{M}}^{(-4)}}{\epsilon^4} + \mathbf{C}_{\mathcal{M}}^{(-1)}\frac{f_\epsilon}{\epsilon} + \mathcal{O}(\epsilon)\right]\mathcal{M}^{(0)}$$

$$\mathbf{C}_{\mathcal{M}}^{(-4)} = \frac{\mathbf{C}_{33}^{(4,-4)}}{2} - \frac{C_A^4}{72} - \frac{1}{6}\frac{d_{AA}}{N_A} + \frac{1}{2}\left(\frac{d_{AR_i}}{N_{R_i}} + \frac{d_{AR_j}}{N_{R_j}}\right)$$

$$\mathbf{C}_{\mathcal{M}}^{(-1)} = \mathbf{C}_{33}^{(4,-1)} + \frac{101\, C_A^4}{36} + \frac{110}{3}\frac{d_{AA}}{N_A} - 104\left(\frac{d_{AR_i}}{N_{R_i}} + \frac{d_{AR_j}}{N_{R_j}}\right)$$

Result holds in every gauge theory. Next steps

- Extract the universal infrared singularities
- Compute the odd amplitude in $\mathcal{N} = 4$

Infrared factorisation

Factorisation of $\hat{\mathcal{M}}$ follows

$$\mathcal{H} = \tilde{\mathbf{Z}}^{-1}\, e^{-H_{1\to 1} L}\, \hat{\mathcal{M}},$$

$\tilde{\mathbf{Z}}$ differs from $\mathbf{Z}$ only by the collinear factors Z_i, Z_j

$$\tilde{\mathbf{Z}} = \mathcal{P}\exp\left[-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\tilde{\mathbf{\Gamma}}\right] \quad \tilde{\mathbf{\Gamma}} = \frac{\gamma_K}{2}\left[L\,\mathbf{T}_t^2 + i\pi\mathbf{T}_{s-u}^2\right] + \mathbf{\Delta}$$

We impose the cancellation of singularities in $\hat{\mathcal{M}}$

- Highest poles dictated by $\gamma_K \to$ check $\hat{\mathcal{M}}$
- Single pole determines $\mathbf{\Delta}$ at 4 loops
- The finite reminder gives $\mathcal{H}$

Infrared singularities

Higher poles

The reduced amplitudes have poles $1/\epsilon^4$

- Highest poles dictated by $\gamma_K \rightarrow$ ✓ check $\hat{\mathcal{M}}$

Four-loop soft anomalous dimension at NNLL

$$\mathrm{Re}\left[\mathbf{\Delta}^{(4,2)}\right] = \zeta_2\zeta_3\mathbf{C}_\Delta$$

$$\mathbf{C}_\Delta = \frac{\mathbf{T}_t^2[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]}{4} + \frac{3}{4}[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2 + \left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\right)$$

- Planar terms in $\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\right)$ cancel
- Manifestly non-planar, new quartic Casimir in $\tilde{\mathbf{\Gamma}}$

Finite parts in $\mathcal{N} = 4$

Finite parts are theory dependent as they involve

- Two-loop impact factors
- $H_{1\to 1}$ to three loops

both determined in (Caron-Huot, Gardi, Vernazza 2017) using three-loop ampliudes (Henn, Mistlberger 2016).

Result

$$\mathrm{Re}\left[\mathcal{H}^{(4,2)}_{\mathcal{N}=4}\right] = \left[\frac{C_A^4}{128}\zeta_3^2 + \frac{3}{16}\zeta_4\zeta_2 \mathbf{C}_\Delta^{(4,2)}\right]\mathcal{M}^{(0)}$$

- Match large N_c limit
- New non-planar term: **proportional** to $\boldsymbol{\Delta}^{(4,2)}$

Conclusions and Outlook

Infrared singularities

Compute $\mathbf{\Gamma}^{(4)}$ in the high-energy limit $\rightarrow \checkmark$

- Constrain the ansatz (Becher,Neubert 2019) $\rightarrow$ Progress
- Regge limit key to bootstrap $\mathbf{\Gamma}^{(3)} \rightarrow \mathbf{\Gamma}^{(4)}$ future work (Almelid,Duhr,Gardi,McLeod,White 2018)

Finite parts

In the Regge limit finite parts are **universal** too.

- Get the complete amplitudes, including $\mathcal{H} \rightarrow \checkmark$

Structure of the Regge limit

The NNLL tower shows triple-Reggeon exchange

- New colour structure, beyond $\mathcal{M}^0 \rightarrow \checkmark$
- Resummation of the NNLL tower $\rightarrow$ future work

Thank you!

Analiticity and signature I

- Dispersion relation (Caron-Huot,Gardi,Vernazza 2017)

$$\mathcal{M}(s,t) = \frac{1}{\pi}\int_0^\infty d\hat{s}\,\frac{D_s(\hat{s},t)}{\hat{s}-s-i0} + \frac{1}{\pi}\int_0^\infty d\hat{u}\,\frac{D_u(\hat{u},t)}{\hat{u}+s+t-i0}$$

D_s and D_u **discontinuities** in the s- and u-channel
D_s and D_u $\rightarrow$ **Real** functions.

- Laplace transform

$$a_s(j,t) = \int_0^\infty d\hat{s}\,D_s(\hat{s},t)\left(\frac{\hat{s}}{-t}\right)^j$$

Follows $(a_s(j,t))^* = a_s(j^*,t)$ and similar for $a_u(j,t)$

$$\mathcal{M}(s,t) = \frac{i}{2}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{dj}{\sin(\pi j)}\left(a_s(j,t)\left(\frac{-s}{-t}\right)^j + a_u(j,t)\left(\frac{s+t}{-t}\right)^j\right)$$

Analiticity and signature II

- Construct amplitudes with definite parity

$$\mathcal{M}^{(\pm)} = \frac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(u,t)\Big)$$

In the high-energy limit (leading power), it gives

$$\mathcal{M}^{(+)} = i\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \cos\left(\frac{\pi j}{2}\right) \left(a_s(j,t) + a_u(j,t)\right) e^{jL},$$

$$\mathcal{M}^{(-)} = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \sin\left(\frac{\pi j}{2}\right) \left(a_s(j,t) - a_u(j,t)\right) e^{jL}.$$

The integrals are **real** because $a_{s/u}(j^*) = \big(a_{s/u}(j)\big)^*$

- $\mathcal{M}^{(+)}$ is **Imaginary**
- $\mathcal{M}^{(-)}$ is **Real**