Feynman integrals, Hypergeometric functions and Mellin-Barnes representation

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Table of Contents

Table of Contents

Motivation
Precision Tests

Precision tests play a central role in the validity of quantum field theory. The two essential ingredients of which are :

- Precise experimental measurement.
- Precise theoretical calculation.

In recent years, much progress has been made in experiments to reduce uncertainty in measured values. For same level of accuracy, this inevitably requires similar precision in theoretical calculations to test the experimental results.

Motivation

Feynman Integrals

The backbone of the theoretical precision calculations has been the evaluation of Feynman integrals. These integrals are the building blocks in the perturbative framework of quantum field theory and are mandatory to calculate scattering amplitudes.

Motivation

Feynman Integrals

The two significant limitations in the evaluation of Feynman diagrams are:

- For higher loop diagrams, the individual diagrams are too complicated to evaluate using standard techniques.
- For higher order in coupling constant, the total number of contributing Feynman integrals increases drastically.

Due to these reasons, development of efficient evaluation techniques has been an active research field in both physics and mathematics community.

Motivation

Feynman Integrals

The two distinct methods to evaluate Feynman integrals are:

- Numerical Methods
  1. Sector Decomposition(A. V. Smirnov 2016)
  2. Method of Regions (B. Ananthanarayan,A. Pal, et al. 2019)
  3. Numerical Integration (T. Hahn 2005)
  4. etc. (V. A. Smirnov, 2013)
- Analytic Methods
  1. Mellin-Barnes Technique (E.E. Boos, A. I. Davydychev 1991)
  2. Intersection Numbers (P. Mastrolia and S. Mizera 2019)
  3. Multiple Polylogarithms (S. Weinzierl 2007)
  4. Yangian Bootstrap Approach (F. Loebbert, D. Müller , et al. 2019)
  5. etc. (V. A. Smirnov, 2013)

In this talk, I will primarily focus on the Mellin-Barnes technique.

Motivation
Multiple Hypergeometric Functions

Hypergeometric functions (H.M.Srivastava. 1985) are special infinite series appearing in almost every branches of physics. The simplest of which is the Gauss hypergeometric function ${}_2F_1(x)$,

$$ {}_2F_1(x) = \sum_{n=0}^{\infty} \frac{\Gamma[a+n]\Gamma[b+n]}{\Gamma[c+n]} \, \frac{x^n}{n!}, \qquad |x| < 1 \qquad (1)$$

Other well known hypergeometric functions include Appell and Lauricella hypergeometric functions.

Motivation
Multiple Hypergeometric Functions

Multiple hypergeometric functions are multi-fold generalizations of the ${}_2F_1$ function. One simple example of this type is the Srivastava's H_C (H.M.Srivastava. 1967) series,

$$H_C = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{\Gamma(a+n_1+n_2)\Gamma(b+n_1+n_3)\Gamma(c+n_2+n_3)}{\Gamma(d+n_1+n_2+n_3)} \frac{u^{n_1} v^{n_2} w^{n_3}}{n_1! n_2! n_3!} \tag{2}$$

valid for $|u|+|v|+|w| < 2 + 2\sqrt{(1-|u|)(1-|v|)(1-|w|)}$.

Motivation

Multiple Hypergeometric Functions

The theory of multiple hypergeometric functions is far from complete. Out of many, two fundamental problems that have no systematic analysis are:

- In principle, using Horn's theory one can find the convergence region of any hypergeometric function but in practice it is too hard to compute.
- Analytic continuations of most three or higher fold hypergeometric functions are unknown.

I will re-address these problems at the end of my talk.

Table of Contents

Theoretical Background
Mellin-Barnes Integral

Mellin-Barnes (MB) integrals are special type of contour integrals introduced in 1888 by Pincherle, later developed by Mellin and Barnes. These integrals primarily contains Euler-Gamma functions in its integrand.
The MB representation of ${}_2F_1$ function is:

$$ {}_2F_1(x) = \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(a+z)\Gamma(b+z)}{\Gamma(c+z)} (-x)^z \qquad (3) $$

where by closing the contour to the right, thereby considering the poles of $\Gamma(-z)$ gives us the usual series representation of ${}_2F_1$.

Theoretical Background

Mellin-Barnes Integral

The characteristic feature of MB integrals is that poles of $\Gamma(\cdots + z)$ must be separated by the contour from poles of $\Gamma(\cdots - z)$. For example, this simple MB integral

$$\int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i}\Gamma(-z)\Gamma(-1/2+z)(-x)^z \tag{4}$$

has poles at $z = 0, 1, 2, \cdots$ and $z = 1/2, -1/2, -3/2, \cdots$ from $\Gamma(-z)$ and $\Gamma(-1/2+z)$ respectively.

Theoretical Background

Mellin-Barnes Integral

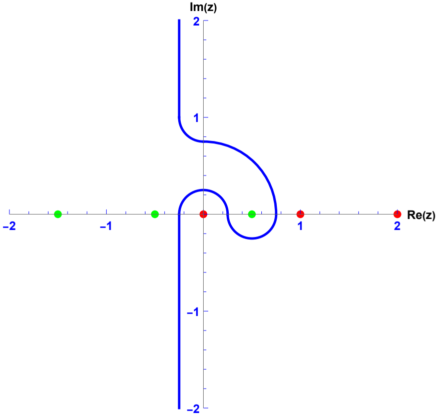


Figure: The contour (in blue) separates the poles of $\Gamma(-z)$ in red from poles of $\Gamma(-1/2 + z)$ in green.

Theoretical Background

Mellin-Barnes Integral

- Closing the contour to the right takes into account the poles at $z = 0, 1, 2..$ so the solution is:

$$\sum_{n=0}^{\infty} \Gamma(-1/2+n)\frac{(x)^n}{n!} \qquad |x| < 1 \tag{5}$$

- Closing to the left gives us,

$$\sum_{n=0}^{\infty} \Gamma(-1/2+n)\frac{(x)^{1/2-n}}{n!} \qquad |x| > 1 \tag{6}$$

Theoretical Background

Mellin-Barnes Integral

A multi-fold MB integral (O. N. Zhdanov, A. K. Tsikh 1998) is of the form :

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} \frac{\prod_{i=1}^{k} \Gamma^{a_i}(\mathrm{e}_i \cdot \mathrm{z} + g_i)}{\prod_{j=1}^{l} \Gamma^{b_j}(\mathrm{f}_j \cdot \mathrm{z} + h_j)} x_1^{z_1} \cdots x_N^{z_N} \tag{7}$$

where a_i, b_j, k, l and N are positive integers. e_i, f_j are N-dimensional coefficient vectors.

Theoretical Background

Mellin-Barnes Integral

There are two types of MB integrals:

- Degenerate Case: Here $\Delta = \sum e_i - \sum f_j = 0$, and several series solutions coexist which are analytic continuations of each other. The type of solution will be hypergeometric.
- Non-Degenerate Case: Here $\Delta = \sum e_i - \sum f_j \neq 0$, and there will be one convergent series converging for all values of parameters. Additionally, asymptotic series also arises.

The conic hull method works for both!

Theoretical Background

Mellin-Barnes Integral

MB integrals can be further classified based on the singularity structure:

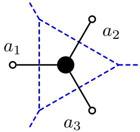
- Non-resonant Case: Here, the number of singular hyper-planes intersecting at any pole is equal to the fold of the MB
- Resonant Case: Here, the number of singular hyper-planes intersecting at some poles can be greater than the fold of the MB

I will consider only the simpler non-resonant case in this talk.

Theoretical Background

Mellin-Barnes Integral

Feynman integrals can be written in terms of MB integrals. For example, conformal 3-point, 1-loop massive Feynman integral:



has the MB representation:

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_2}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_3}{2\pi i} (-u)^{z_1} (-v)^{z_2} (-w)^{z_3} \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3) \times \frac{\Gamma(a_1+z_1+z_2)\Gamma(a_2+z_1+z_3)\Gamma(a_3+z_2+z_3)}{\Gamma(D/2+1/2+z_1+z_2+z_3)}$$

Theoretical Background
Mellin-Barnes Integral

- Both Feynman integrals and Hypergeometric functions can be written in terms of MB integrals.
- The simplest technique to convert a Feynman integral to a MB integral requires repetitive application of

$$\frac{1}{(A+B)^{\alpha}} = \frac{1}{\Gamma(\alpha)} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \Gamma(-z)\Gamma(\alpha+z) A^{-\alpha-z} B^{z} \tag{8}$$

  on the momentum representation of Feynman integrals.
- The Mathematica based package AMBRE (J. Gluza, K. Kajda, T. Riemann 2007) automatizes the derivation of MB integrals for Feynman integrals.

Theoretical Background

Mellin-Barnes Integral

Current status of evaluating MB integrals:

- One-fold MB: By closing the contour to the right or left.
- Two-fold MB: Developed by A. Tsikh et al.(M. Passare, A. K. Tsikh, O. N. Zhdanov 1994) and generalized by S. Friot and D. Greynat(S. Friot, D. Greynat 2011).
- Three and higher-fold MB: This was a 100-year-old problem, which we have solved using conic hull.

Theoretical Background

Conic Hull

Conic hulls are semi-infinite geometric regions. The parametric representation of n-dimensional conic hull is given by:

$$\{p + s_1 v_1 + \cdots + s_n v_n | s_i \in \mathcal{R}\} \tag{9}$$

where, the point p is the vertex and v_i's are the basis vectors. For example, if $p = (0,0)$ and $v_1 = (1,0), v_2 = (1,1)$

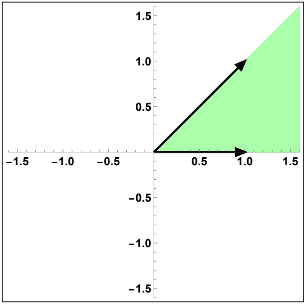


Table of Contents

Solution
Previous Approaches

- Since the beginning of the extensive use of MB integrals there were several attempts to solve N-fold case or related problem.
- One recent approach is the Yangian bootstrap approach using symmetry of the Feynman integrals and underlying field theory.
- This method is able to extract building blocks but not the MB solution for complicated case, as this requires the convergence region of building blocks, which are generally hard to compute.

Solution
Our Approach

The first part of our approach is to find all the building blocks. To illustrate, we consider the following 2-fold MB integral,

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_2}{2\pi i} (-u_1)^{z_1} (-u_2)^{z_2} \Gamma(-z_1)\Gamma(-z_2) \times \frac{\Gamma(a+z_1+z_2)\Gamma(b_1+z_1)\Gamma(b_2+z_2)}{\Gamma(c+z_1+z_2)} \tag{10}$$

associated with Appell F_1 series (up-to an overall factor).

Solution
Our Approach

We begin by tabulating the coefficient vectors of all gamma functions in the numerator:

i	Γ function	e_i
1	$\Gamma(-z_1)$	$(-1, 0)$
2	$\Gamma(-z_2)$	$(0, -1)$
3	$\Gamma(a + z_1 + z_2)$	$(1, 1)$
4	$\Gamma(b_1 + z_1)$	$(1, 0)$
5	$\Gamma(b_2 + z_2)$	$(0, 1)$

Solution
Our Approach

- We next consider all possible two-combinations of numerator gamma function denoted by K_{i_1,i_2}, where i_1 and i_2 are the labels of the gamma functions in the two-combination. For example, $K_{1,3}$ denotes $\{\Gamma(-z_1), \Gamma(a+z_1+z_2)\}$.
- So there are $\binom{5}{2} = 10$ possible two-combinations for the Appell F_1's MB.
- We retain only those two-combinations for which the associated matrix $A_{i_1,i_2} = (e_{i_1}, e_{i_2})^T$ is non-singular.

Solution
Our Approach

- The following are the retained two-combinations for the Appell F_1's MB,

$$\left\{K_{1,2}\,,K_{1,3}\,,K_{1,5}\,,K_{2,3}\,,K_{2,4}\,,K_{3,4}\,,K_{3,5}\,,K_{4,5}\right\} \tag{11}$$

Only 8 out of $\binom{5}{2} = 10$ possible two-combinations are retained.

- For example, the two-combination $K_{1,4} = \{\Gamma(-z_1), \Gamma(b_1 + z_1)\}$ is omitted as the corresponding matrix $A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$ is singular.

Solution

Our Approach

- We then associate a series, denoted by B_{i_1,i_2} with each retained two-combination
- This series is obtained by adding the residues of only those poles formed by the intersection of singular hyper-planes of gamma functions in the two-combination, divided by $|det(A_{i_1,i_2})|$.

Solution
Our Approach

For example, the poles of $K_{1,3} = \{\Gamma(-z_1), \Gamma(a+z_1+z_2)\}$ are at $(z_1, z_2) = (n_1, -a-n_1-n_2)$ for $n_i \geq 0$. Therefore,

$$B_{1,3} = (-u_2)^{-a} \sum_{n_1,n_2=0}^{\infty} \left(-\frac{u_1}{u_2}\right)^{n_1} \left(\frac{1}{u_2}\right)^{n_2} \times \frac{\Gamma(a+n_1+n_2)\,\Gamma(b_1+n_1)\,\Gamma(-a+b_2-n_1-n_2)}{\Gamma(n_1+1)\,\Gamma(n_2+1)\,\Gamma(-a+c-n_2)} \tag{12}$$

Solution

Our Approach

- As there are 8 retained two-combinations so we have total 8 series, which we term as building blocks, which in our case are hypergeometric functions.
- Building blocks are not the solutions of the MB integral.
- The solution of the MB are linear combinations of the building blocks.

Solution
Our Approach

- The general solution of an MB integral can be written as:

$$\sum_{\alpha \in S} c_\alpha B_\alpha \tag{13}$$

where the set S contains all the labels (i_1, i_2) of building blocks B_{i_1,i_2}.

- Each c_α's are either 0 or 1.
- Each combination of c_α corresponds to a solution of the MB integral.
- The building blocks can therefore be thought as the basis for building MB solutions.

Solution
Our Approach

- The hardest part in the evaluation of MB integrals is to find the coefficients c_α.
- In the previous approaches, this requires finding the convergence regions of each building blocks B_{i_1,i_2}.
- As these building blocks are multi-fold hypergeometric functions whose convergence regions are mostly unknown, the previous approaches failed.

Solution
Our Approach

The novelty of our conic hull approach is to bypass the computation of convergence region of the building blocks.

Introducing Conic Hulls

- Assign a conic hull to each building block B_{i_1,i_2} denoted by C_{i_1,i_2}, whose edges are parallel to the vectors e_{i_1} and e_{i_2} with vertex at the origin.
- A solution of MB is obtained by summing the building blocks associated with the largest subset of conic hulls having a common intersection.

Solution

Our Approach

- The convergence region of a MB solution is equal to the common convergence region of it's building blocks, which are themselves difficult to compute.
- To partially bypass this we also introduce the concept of master series which is obtained by considering the common intersection conic hull (master conic hull) and mapping this back to a series.
- We conjecture that the convergence region of the master series, will coincide with the convergence region of the series representation thereby avoiding the analysis of individual building blocks.

Solution
Our Approach

Heart of the Method

Building Blocks $\leftrightarrow$ Conic hull
MB Solution $\leftrightarrow$ Intersection of Conic hulls
Master Series $\leftrightarrow$ Common intersection region of Conic hulls

The problem of evaluating MB is therefore reduced to analyzing conic hulls.

Solution
Our Approach

8 conic hulls for each building blocks associated with Appell F_1.

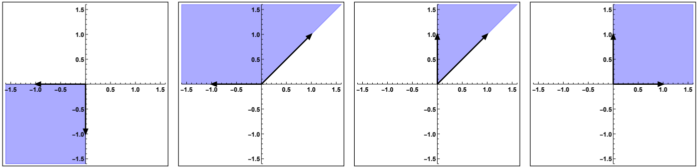


Figure: Conic hulls of: $B_{1,2}, B_{1,3}, B_{3,5}, B_{4,5}$ from left to right

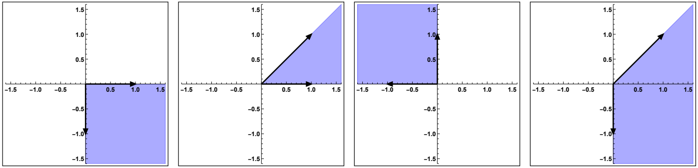
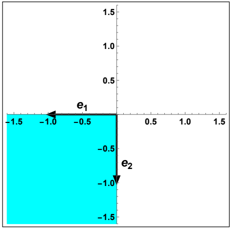


Figure: Conic hulls of: $B_{2,4}, B_{3,4}, B_{1,5}, B_{2,3}$ from left to right

Solution
Our Approach

Consider the conic hull associated with $B_{1,2}$ built from the poles of $\{\Gamma(-z_1), \Gamma(-z_2)\}$, with edges along $e_1 = (-1, 0)$ and $e_2 = (0, -1)$.

$$B_{1,2} = \sum_{n_1, n_2 = 0}^{\infty} \frac{\Gamma(a + n_1 + n_2)\Gamma(b_1 + n_1)\Gamma(b_2 + n_2)}{\Gamma(c + n_1 + n_2)} \frac{u_1^{n_1} v_2^{n_2}}{n_1! n_2!} \tag{14}$$



This conic hull does not intersect with other conic hulls, so $B_{1,2}$ itself is a solution of the MB.

Solution
Our Approach

Next, consider the conic hulls associated with three building blocks $B_{1,3}$, $B_{3,5}$ and $B_{4,5}$ of the Appell F_1 series:

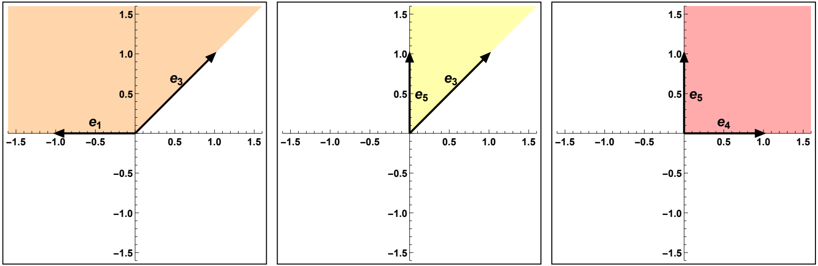
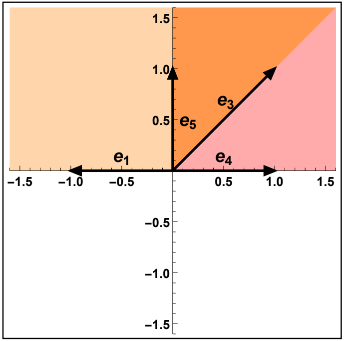


Figure: Conic hulls $C_{1,3}$ (left), $C_{3,5}$ (center) and $C_{4,5}$ (right).

Solution
Our Approach

The conic hulls $C_{1,3}$, $C_{3,5}$ and $C_{4,5}$ have a common intersection. Thus, one of the MB solution is $B_{1,3} + B_{3,5} + B_{4,5}$.



The common intersecting conic hull coincide with the conic hull $C_{3,5}$, so $B_{3,5}$ is the master series for this MB solution.

Solution

Our Approach

A straightforward analysis of all the 8 conic hulls gives us 5 different MB solution.

$$
= \begin{cases}
B_{1,2} & \text{for } |u_1| < 1 \cap |u_2| < 1 \quad (\mathcal{R}_1) \\
B_{1,3} + B_{3,5} + B_{4,5} & \text{for } \left|\frac{1}{u_1}\right| < 1 \cap \left|\frac{u_1}{u_2}\right| < 1 \quad (\mathcal{R}_3) \\
B_{1,3} + B_{1,5} & \text{for } |u_1| < 1 \cap \left|\frac{1}{u_2}\right| < 1 \quad (\mathcal{R}_2) \\
B_{2,3} + B_{2,4} & \text{for } \left|\frac{1}{u_1}\right| < 1 \cap |u_2| < 1 \quad (\mathcal{R}_4) \\
B_{2,3} + B_{3,4} + B_{4,5} & \text{for } \left|\frac{u_2}{u_1}\right| < 1 \cap \left|\frac{1}{u_2}\right| < 1 \quad (\mathcal{R}_5)
\end{cases} \tag{15}
$$

Solution
Our Approach

Convergence region of the 5 series representation for the Appell F_1 series,

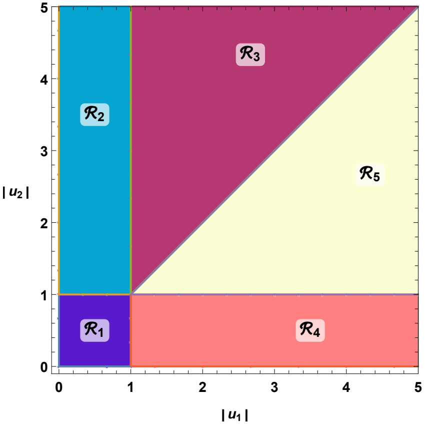
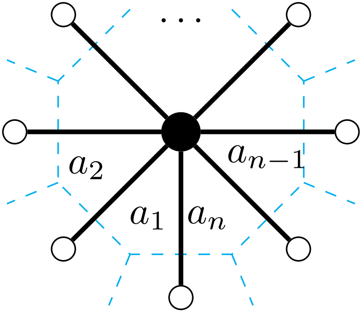


Table of Contents

Conclusion

New Results

A recent conjecture was made based on the Yangian bootstrap analysis (F. Loebbert, J. Miczajka , et al. 2020), which states that dual-conformal n-point one-loop Feynman integrals can be written as a single series representation for two set of conformal variables.



Conclusion

New Results

Set 1 : The corresponding $\frac{n(n-1)}{2}$ MB-representation can be written as:

$$I_{n\bullet}^{m_1\ldots m_n} = \frac{\pi^{D/2+1/2}}{2^{D-1}\prod_{i=1}^{n}\Gamma(a_i)m_i^{a_i}} \prod_{\alpha\in B_n}\left(\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}\,z_\alpha}{2\pi i}\Gamma(-z_\alpha)(-u_\alpha)^{z_\alpha}\right) \times \frac{\prod_{i=1}^{n}\Gamma\left(a_i+\sum_{\alpha\in B_{n|i}} z_\alpha\right)}{\Gamma\left(\frac{D+1}{2}+\sum_{\alpha\in B_n} z_\alpha\right)} \tag{16}$$

where $B_n = \{12, 13, 23, \cdots, (n-1, n)\}$ is the set of pairs of distinct integers (written in increasing order) in $\{1, \cdots, n\}$ and $B_{n|j}$ is the subset of B_n with pairs containing j.

Conclusion

New Results

We prove this conjecture by applying the conic hull theory on the MB representation.

Proof

- The conic hull associated with the $\frac{n(n-1)}{2}$-combination consisting of the gamma functions $\Gamma(-z_\alpha)$ belongs to the $(-, ..., -)$ hyper-quadrant.
- The coefficient vectors of all the remaining numerator gamma functions lies in the $(+, ..., +)$ hyper-quadrant, hence, all other conic hulls cannot lie in the $(-, ..., -)$ hyper-quadrant.
- Therefore, the trivial conic hull do not intersect with other conic hulls, so its associated building block itself forms a solution, which is a single series.

Conclusion
New Results

Set 2 : The corresponding $\frac{n(n-1)}{2}$ MB-representation can be written as:

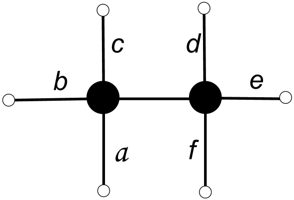
$$I_{n\bullet}^{m_1\ldots m_n} = \frac{\pi^{D/2}}{2^{n-1}\prod_{i=1}^{n}\Gamma(a_i)m_i^{a_i}}\prod_{\alpha\in B_n}\left(\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}\,z_\alpha}{2\pi i}\Gamma(-z_\alpha)(2v_\alpha)^{z_\alpha}\right)$$
$$\times\prod_{i=1}^{n}\Gamma\left(\hat{a}_i+\sum_{\alpha\in B_{n|i}}\hat{z}_\alpha\right) \tag{17}$$

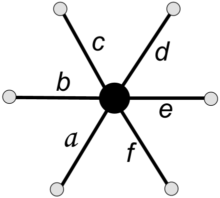
where $\hat{a}_i \doteq a_i/2$.
A similar reasoning as in the previous slide gives the proof of the conjecture.

Conclusion

New Results

We computed the previously unsolved dual-conformal fishnet Double-Box and Hexagon diagrams,





both of which has a nine-fold MB representation.

Conclusion
New Results

- The total number of building blocks for Double-Box and Hexagon are 4834 and 2530, respectively. Each of them being a nine-fold hypergeometric series.
- There was an attempt to solve them using the Yangian bootstrap approach but it failed due to large number and poor understanding of the convergence analysis of building blocks.

Conclusion
New Results

- We solved it using the conic hull approach and obtained solutions which are sum of 44 and 26 building blocks for Double-Box and Hexagon, respectively.
- We also obtained several other solutions which are analytic continuations of each other.

Conclusion

What is the progress?

- Most Feynman integrals can be written in terms of MB integrals. Applying the conic hull theory will then straightforwardly give series representation without any convergence analysis.
- The numerical computation is much faster with the series representation than numerical integration. For example, numerical integration of hexagon takes 9 hours, whereas it takes only 3 mins to numerically sum, for the same level of accuracy.

Conclusion

What is the progress?

- Most hypergeometric functions can also be written in terms of MB integrals. The conic hull theory will then give us several series representations, which are analytic continuations of the original hypergeometric series.
- Thus, the conic hull theory provides a systematic procedure to obtain various analytical continuations.

Conclusion

What is the progress?

Our method is superior to the Yangian bootstrap approach for the following reasons:

- We bypass convergence region of building blocks to form the MB solution.
- Our method can also be applied to hypergeometric functions.
- We can find overall constant factors of building blocks analytically.
- Our method can be applied to both non-resonant and resonant case.

Conclusion
Future Work

- To build an automatized code.
- To compare with MBsums (M. Ochman, T. Riemann 2015).
- For conformal one-box and simpler diagrams, it was shown that linear combinations can be completely fixed by Yangian symmetry. But it failed for the double-box and hexagon case. Thus, it remains to see if there are additional/hidden symmetries which can fix the linear combination by exploring the interplay between conic hull theory and Yangian bootstrap.
- For higher fold MB the series representation are bulky, hence developing an algorithmic approach to write them in closed form in terms of multiple polylogarithms or elliptic polylogarithms, will be an important achievement.