On Singularity Resolutions, Evaluations and Reductions of Feynman Integrals

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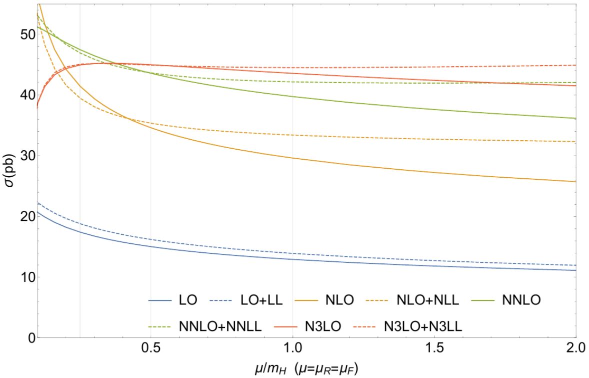
IIT Hyderabad HEP Seminar
April 4, 2018

Higgs at N^{3}LO and resummations



[Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger, Lazopoulos '16]

- plot using approximate N^{3}LO, important: subleading terms in threshold expansion
- exact N^{3}LO [Mistlberger '18] in excellent agreement (not so much for subleading partonic channels)
- resummation improves convergence of perturbative expansion
- missing for N^{3}LL: cusp anomalous dimension @ 4 loops !

Towards the cusp anomalous dimension @ 4-loops

Cusp anomalous dimension @ 4-loops:

- required for N^3LL resummation
- Casimir scaling for quark and gluon cusp anomalous dimension:

$$\boxed{\Gamma_4^q \overset{?}{=} \frac{C_F}{C_A} \Gamma_4^g}$$

- partial results: [Grozin, Henn, Korchemsky, Marquard '15], [Ruijl, Ueda, Vermaseren, Davies, Vogt '16]
- numerical result for cusp in $\mathcal{N} = 4$ SYM: [Boels, Hubert, Yang '17]
- (numerical) result for quark cusp: [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]

4-loop form factors:

- $1/\epsilon^2$ poles allow extraction of cusp anomalous dimension
- reduced integrand for $\mathcal{N} = 4$ SYM: [Boels, Kniehl, Tarasov, Yang '12, '15]
- leading N_c quark F_4^q: [Henn, Smirnov, Smirnov, Steinhauser, Lee '16, '16]
- n_f^3 quark F_4^q and gluon F_4^g: [Manteuffel, Schabinger '16]
- n_f^2 quark F_4^q: [Lee, Smirnov, Smirnov, Steinhauser, Lee '17]

this talk: QCD form factors via finite integrals and finite fields

1. basis of finite integrals
2. reductions via finite fields
3. first results

Part I: A basis of finite Feynman integrals
(singularity resolution and evaluation)

[AvM, Panzer, Schabinger]

Multi-Loop Feynman integrals

$$I = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \qquad a_i \in \mathbb{Z}, \quad D_1 = k_1^2 - m_1^2 \text{ etc.}$$

family of loop integrals:

- fulfill linear relations: integration-by-parts identities
- systematic reduction to master integrals possible
- think of it as linear vector space with some finite basis
- specific basis choices:
  - canonical basis for method of differential equations [Henn '13]
  - basis of finite integrals for direct integration (analyt., numeric.): this talk

An improved basis for Feynman parameters

consider Feynman parameter representation of multi-loop integral

$$I = N\left[\prod_{j=1}^{N}\int_0^{\infty} \mathrm{d}x_j x_j^{\nu_k - 1}\right]\delta(1-x_N)\,\mathcal{U}^{\nu-(L+1)\frac{d}{2}}\,\mathcal{F}^{-\nu+L\frac{d}{2}}$$

where

- $\nu = \sum_i \nu_i$, ν_i denotes propagator multiplicity
- $\mathcal{U}$ and $\mathcal{F}$ are Symanzik polynomials in x_i

problem:

- can't directly expand in $\epsilon = (4-d)/2$: divergencies from x_i integrations
- no straight-forward analytical or numerical integration

generic approaches to singularity resolution:

1. sector decomposition [Hepp '66, Binoth, Heinrich '00]
2. polynomial exponent raising [Bernstein '72, Tkachov '96, Passarino '00]
3. analytic regularisation [Panzer '14]
4. basis of finite Feynman integrals ("dims & dots") [AvM, Schabinger, Panzer '14]

Sector decomposition

- very established method + codes
- but not always ideal: for example, calculate to $\mathcal{O}(\epsilon)$:

$$I(\epsilon) = \int_0^1 \mathrm{d}t\; t^{-1-\epsilon}(1-t)^{-1-2\epsilon}{}_2F_1(\epsilon, 1-\epsilon; -\epsilon; t)$$

decompose into sectors: split at (arbitrary) $t = 1/2$, rescale, expand in plus distributions:

$$I_1(\epsilon) = -\frac{1}{\epsilon} - 1 + \left(3 + \frac{1}{3}\pi^2 - 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

$$I_2(\epsilon) = -\frac{1}{3\epsilon} + \frac{7}{3} + \left(-7 + \frac{1}{3}\pi^2 + 8\ln(2)\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\,.$$

result:

$$I(\epsilon) = -\frac{4}{3\epsilon} + \frac{4}{3} + \left(-4 + \frac{2}{3}\pi^2\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\,.$$

split up of domain introduces spurious terms ln(2)

- can be worse: spurious order 5 polynomial denominators: [AvM, Schabinger, Zhu '13]
- destroys linear reducibility: no analytical integration a la [Brown '08; Panzer '14; Bogner '15]

Analytic regularisation [Panzer '14]

Euclidean integrals: all subdivergencies from integration boundaries

- check: rescale $x_j \to \lambda x_j$ or x_j/λ for some $j \in J$
- problematic scaling of integrand for $\lambda \to 0$ signals divergency
- convergence can be improved by regularising trafo based on partial integration: new integrand

$$P' = -\frac{1}{\omega_J(P)} \frac{\partial}{\partial \lambda} \lambda^{-\deg_J(P)} P_{J_\lambda} \Big|_{\lambda \to 1} .$$

  iterate if necessary
- maps original integral to sum of dimensionally shifted integrals with higher powers of propagators (dots)

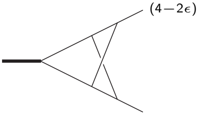
shortcomings:

- proliferation of terms, ambiguities

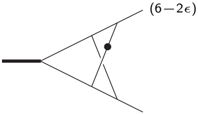
way out:

- consider full set of master integrals (basis)
- employ integration by parts (IBP) reductions

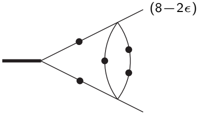
observation: always possible to decompose wrt basis of finite integrals



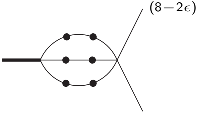
$$= -\frac{4(1-4\epsilon)}{\epsilon(1-\epsilon)q^2}$$



$$-\frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4(1-\epsilon)^2(2-\epsilon)^2 q^2}$$



$$+\frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)(1-\epsilon)^2(2-\epsilon)^2 q^2}$$



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basis consists of standard Feynman integrals, but

- in shifted dimensions
- with additional dots (propagators taken to higher powers)
- much more compact than old reg. shifts

Practical algorithm for basis construction

given the existence proof, forget about previous construction and just do:

Algorithm: construction of finite basis

- systematic scan for finite integrals with dim-shifts and dots
- IBP + dimensional recurrence for actual basis change

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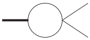
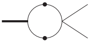
remarks:

- computationally expensive part shifted to IBP solver
- efficient, easy to automate
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon

Form factors @ 1-loop

- consider one-loop quark and gluon form factors in massless QCD
- integral basis change to finite integrals

$(4-2\epsilon)$

$$= \frac{1}{\epsilon(1-\epsilon)}$$
$(6-2\epsilon)$


dot: squared propagator, subscript: space-time dimension

Form factors @ 1-loop

- consider one-loop quark and gluon form factors in massless QCD
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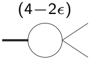
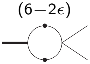


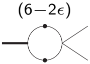
$$= \frac{1}{\epsilon(1-\epsilon)}$$



dot: squared propagator, subscript: space-time dimension

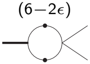
- form factors

$$\mathcal{F}_1^q(\epsilon) = C_F \, \frac{1}{\epsilon^2} \, a_1$$



$$a_1 = \frac{-2+\epsilon-2\epsilon^2}{1-\epsilon}$$

$$\mathcal{F}_1^g(\epsilon) = C_A \, \frac{1}{\epsilon^2} \, b_1$$



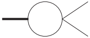
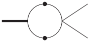
,

$$b_1 = \frac{-2(1-3\epsilon+2\epsilon^2+\epsilon^3)}{(1-\epsilon)^2}$$

note: all divergencies explicit

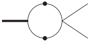
Form factors @ 1-loop

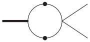
- consider one-loop quark and gluon form factors in massless QCD
- integral basis change to finite integrals

(4−2ϵ)  $= \dfrac{1}{\epsilon(1-\epsilon)}$ (6−2ϵ) 

dot: squared propagator, subscript: space-time dimension

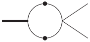
- form factors

$\mathcal{F}_1^q(\epsilon) = C_F \, \dfrac{1}{\epsilon^2} \, a_1$ (6−2ϵ)  $\qquad a_1 = \frac{-2+\epsilon-2\epsilon^2}{1-\epsilon}$

$\mathcal{F}_1^g(\epsilon) = C_A \, \dfrac{1}{\epsilon^2} \, b_1$ (6−2ϵ) , $\qquad b_1 = \frac{-2(1-3\epsilon+2\epsilon^2+\epsilon^3)}{(1-\epsilon)^2}$

note: all divergencies explicit

- expansion in ϵ

(6−2ϵ)  $= 1 + \epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3)$

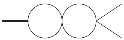
$$a_1 = -2 - \epsilon - 3\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$$b_1 = -2 + 2\epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3)$$

- Casimir scaling reflected by $a_1|_{\epsilon=0} = b_1|_{\epsilon=0}$

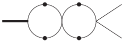
Form factors @ 2-loops: to finite basis

$(4-2\epsilon)$



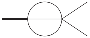
$= \frac{1}{\epsilon^2}\frac{1}{(1-\epsilon)^2}$

$(6-2\epsilon)$



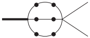
,

$(4-2\epsilon)$



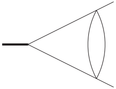
$= \frac{1}{\epsilon}\frac{-4}{(2-\epsilon)^2(1-\epsilon)^2(1-2\epsilon)}$

$(8-2\epsilon)$



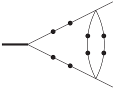
,

$(4-2\epsilon)$



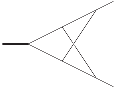
$= \frac{1}{\epsilon^2}\frac{16(3-2\epsilon)(2-3\epsilon)}{(3-\epsilon)^2(2-\epsilon)^2(1-\epsilon)^3(1+2\epsilon)}$

$(10-2\epsilon)$



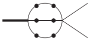
,

$(4-2\epsilon)$



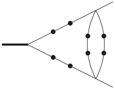
$= \frac{1}{\epsilon^4}\frac{-4(2-3\epsilon)(14-81\epsilon+115\epsilon^2+14\epsilon^3-132\epsilon^4+72\epsilon^5)}{(2-\epsilon)^2(1-\epsilon)^2(1-2\epsilon)^2(2-\epsilon-2\epsilon^2)}$

$(8-2\epsilon)$



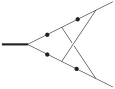
$+ \frac{1}{\epsilon^4}\frac{-16(1+\epsilon)(3-2\epsilon)(2-3\epsilon)(10-61\epsilon+102\epsilon^2-44\epsilon^3-8\epsilon^4)}{(3-\epsilon)^2(2-\epsilon)^2(1-\epsilon)^3(1-2\epsilon)(1+2\epsilon)(2-\epsilon-2\epsilon^2)}$

$(10-2\epsilon)$

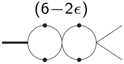


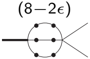
$+ \frac{1}{\epsilon}\frac{4(3-4\epsilon)(1-4\epsilon)}{(2-\epsilon)(1-\epsilon)(1-2\epsilon)(2-\epsilon-2\epsilon^2)}$

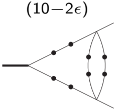
$(8-2\epsilon)$

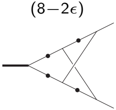


quark form factor

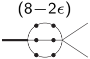
$\mathcal{F}_2^q(\epsilon) = C_F^2 \Bigg\{ \frac{1}{\epsilon^4} \Bigg[c_1$ 

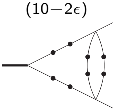
$+ c_2$ 

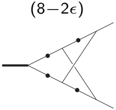
$\Bigg] + \frac{1}{\epsilon^3} \Bigg[c_3$ 

$\Bigg] + \frac{1}{\epsilon} \Bigg[c_4$ 

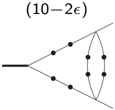
$\Bigg] \Bigg\}$

$+ C_F C_A \Bigg\{ \frac{1}{\epsilon^4} \Bigg[c_5$ 

$+ c_6$ 

$\Bigg] + \frac{1}{\epsilon} \Bigg[c_7$ 

$\Bigg] \Bigg\}$

$+ C_F N_f \Bigg\{ \frac{1}{\epsilon^3} \Bigg[c_8$ 

$\Bigg] \Bigg\}$

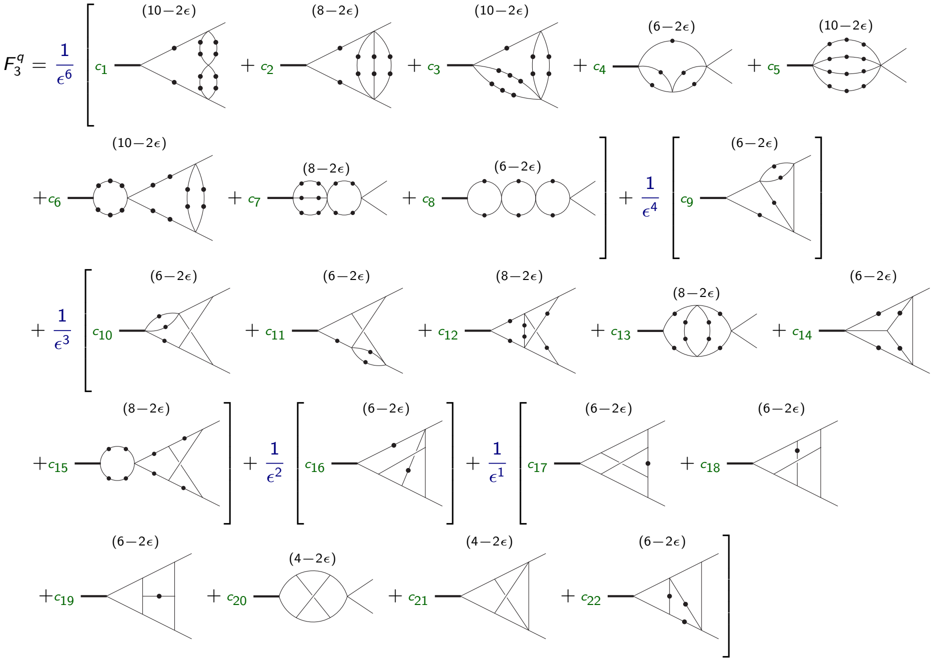
Form factors @ 3-loops

- master integrals:
  - [Gehrmann, Heinrich, Huber, Studerus '06]
  - [Heinrich, Huber, Maître '07]
  - [Heinrich, Huber, Kosower, V. Smirnov '09]
  - [Lee, A. Smirnov, V. Smirnov '10]
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Lee, V. Smirnov '10] $\Leftarrow$ the only complete weight 8
  - [Henn, A. Smirnov, V. Smirnov '14] (diff. eqns.)

- form factors @ 3-loops:
  - [Baikov, Chetyrkin, A. Smirnov, V. Smirnov, Steinhauser '09]
  - [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10, '10]

- recalculation of 3-loop results via finite integrals:
  - [AvM, Panzer, Schabinger '15]
  - automated setup, fully analytical
  - `Qgraf` [Nogueira]:
    - Feynman diagrams
  - `Reduze 2` [AvM, Studerus]:
    - interferences
    - IBP reductions
    - finite integral finder
    - basis change with dimensional recurrences
  - `HyperInt` [Panzer]:
    - integration of ϵ expanded master integrals

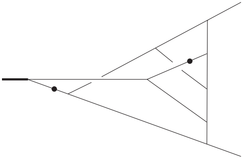
Quark form factor @ 3-loops [AvM, Panzer, Schabinger '15]



Analytical integration @ 4-loops

[AvM, Panzer, Schabinger '15]

a non-planar 12-line topology @ 4-loops:



$$= \frac{18}{5}\zeta_2^2\zeta_3 - 5\zeta_2\zeta_5 + \Big(24\zeta_2\zeta_3 + 20\zeta_5 - \frac{188}{105}\zeta_2^3 - 17\zeta_3^2 + 9\zeta_2^2\zeta_3$$

$$- 47\zeta_2\zeta_5 - 21\zeta_7 + \frac{6883}{2100}\zeta_2^4 + \frac{49}{2}\zeta_2\zeta_3^2 + \frac{1}{2}\zeta_3\zeta_5 - 9\zeta_{5,3}\Big)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

- only shallow ϵ expansion needed
- numerical result with `Fiesta` [A. Smirnov]: straight-forward confirmation
- starts at weight 7, not expected to contribute to cusp anomalous dimension

Numerical evaluations

advantages of (quasi-)finite basis:

- straight-forward to integrate numerically (in principle)
- no cancellation of spurious singularities
- no blow up in number of sectors
- very simple integrands also at high orders in ϵ

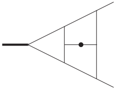
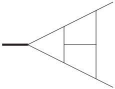
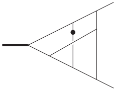
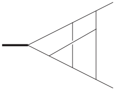
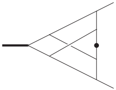
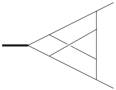
experiments with numerical evaluations:

- naive straight-forward implementation possible but not ideal
- better: employ existing sector decomposition programs
  - `Fiesta` [A. Smirnov]
  - `SecDec` [Borowka, Heinrich, Jones, Kerner, Schlenk, Zirke]
  - `sector_decomposition` [Bogner, Weinzierl]
- used for *HH* @ NLO [Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Schubert, Zirke '16]
- finite integrals: faster & more reliable

Numerical performance

[AvM, Schabinger '17]

improvement wrt conventional basis:

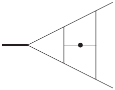
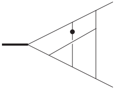
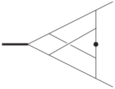
finite	time	rel. err.	conventional	time	rel. err.
$(6-2\epsilon)$ 	128 s	5.12×10^{-6}	$(4-2\epsilon)$ 	39094 s	9.91×10^{-4}
$(6-2\epsilon)$ 	192 s	2.68×10^{-6}	$(4-2\epsilon)$ 	19025 s	9.38×10^{-5}
$(6-2\epsilon)$ 	127 s	2.26×10^{-6}	$(4-2\epsilon)$ 	19586 s	1.07×10^{-4}

timings with Fiesta 4, ϵ expansion through to weight 6

Numerical performance

[AvM, Schabinger '17]

ϵ expansions to high weights feasible:

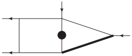
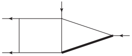
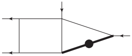
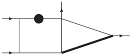
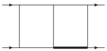
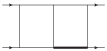
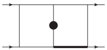
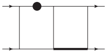
	weight 6		weight 8	
	time	rel. err.	time	rel. err.
$(6-2\epsilon)$ 	128 s	5.12×10^{-6}	491 s	2.22×10^{-5}
$(6-2\epsilon)$ 	192 s	2.68×10^{-6}	761 s	5.84×10^{-6}
$(6-2\epsilon)$ 	127 s	2.26×10^{-6}	485 s	8.45×10^{-6}

timings with Fiesta 4

Numerical performance

[AvM, Schabinger '17]

basis of finite integrals renders problematic double boxes numerically accessible

finite	time	rel. err.	conventional	time	rel. err.
$(6-2\epsilon)$  (s,t)	201 s	2.34×10^{-4}	$(4-2\epsilon)$  (s,t)	384 s	8.12×10^{-4}
$(6-2\epsilon)$  (s,t)	150 s	4.83×10^{-4}	$(4-2\epsilon)$  (s,t)	56538 s	1.67×10^{-2}
$(6-2\epsilon)$  (s,t)	280 s	1.00×10^{-3}	$(4-2\epsilon)$  (s,t)	214135 s	8.29×10^{-3}
$(6-2\epsilon)$  (s,t)	294 s	1.21×10^{-3}	$(4-2\epsilon)$  (s,t)	3484378 s	30.9

timings with SecDec 3 in physical region

Part II: A finite field approach to integral reduction

[AvM, Schabinger]

Integration-by-parts (IBP) identities

in dimensional regularisation, integral over total derivative vanishes:

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(k_j^\mu \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right)$$

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(p_j^\mu \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right)$$

where p_j are external momenta, $a_i \in \mathbb{Z}$, $D_1 = k_1^2 - m_1^2$ etc.

integral reduction:

- express arbitrary integral for given problem via few basis integrals
- integration-by-parts (IBP) reductions [Chetyrkin, Tkachov '81]
- public codes: `Air` [Anastasiou], `Fire` [Smirnov], `Reduze 1` [Studerus], `Reduze 2` [AvM, Studerus], `LiteRed` [Lee]
- possible: exploit structure at algebra level
- here: Laporta's approach

Laporta's algorithm:

1. index integrals by propagator exponents: $I(a_1, \ldots, a_N)$
2. define ordering (e.g. fewer denominators means simpler)
3. generate IBPs for explicit values $a_1, \ldots, a_N$
4. results in linear system of equations
5. solve linear system of equations

major shortcomings of traditional Gauss solvers:

- suffers from intermediate expression swell
- requires large number of auxiliary integrals and equations
- limited possbilities for parallelisation

IBP reductions from finite field samples

A novel approach to IBPs [AvM, Schabinger '14]

1. finite field sampling
   - set variables to integer numbers
   - consider coefficients modulo a prime field $\mathbb{Z}_p$
2. solve finite field system
3. reconstruct rational solution from many such samples

IBP reductions from finite field samples

A novel approach to IBPs [AvM, Schabinger '14]

1. finite field sampling
   - set variables to integer numbers
   - consider coefficients modulo a prime field $\mathbb{Z}_p$
2. solve finite field system
3. reconstruct rational solution from many such samples

finite field techniques:

- no intermediate expression swell by construction
- early discard of redundant and auxiliary quantities
- big potential for parallelisation

established in math literature, becomes popular in physics:

- dense solver: [Kauers]
- filtering: `Ice` [Kant '13]
- tensor reduction: [Heller]
- QCD integrand construction: [Peraro '16]
- symbol algebra: [Dixon, Drummond, Harrington, McLeod, Papathanasiou, Spradlin '16]

core algorithm:

Extended Euclidean Algorithm (EEA)

1. begin with $(g_0, s_0, t_0) = (a, 1, 0)$ and $(g_1, s_1, t_1) = (b, 0, 1)$,
2. then repeat

$$q_i = g_{i-1} \text{ quotient } g_i$$
$$g_{i+1} = g_{i-1} - q_i g_i$$
$$s_{i+1} = s_{i-1} - q_i s_i$$
$$t_{i+1} = t_{i-1} - q_i t_i$$

3. until $g_{k+1} = 0$ for some k. at that point:

$$s_k a + t_k b = g_k = \text{GCD}(a, b)$$

restrict first to linear systems with rational numbers coefficients

- use EEA to define inverse of integer b modulo m with $\text{GCD}(m, b) = 1$:

$$1 = s\, m + t\, b$$
$$\Rightarrow 1/b := t \mod m$$

this gives us a canonical homomorphism ϕ_m of $\mathbb{Q}$ onto $\mathbb{Z}_m$ with

$$\phi_m(a/b) = \phi_m(a)\phi_m(1/b)$$

- for large enough m, the map ϕ_m can be inverted !

given a finite field image of a/b modulo m for $m > 2\max(a^2, b^2)$,
a unique rational reconstruction is possible:

Rational reconstruction [Wang '81; Wang, Guy, Davenport '82]

to reconstruct a/b from its finite field image $u = a/b \mod m$:

- run EEA for u and m
- stop at first g_j with $|g_j| \leq \lfloor\sqrt{m/2}\rfloor$
- the unique solution is $a/b = g_j/t_j$

important details:

- since we don't know bound on m:
  veto $|t_j| > \lfloor\sqrt{m/2}\rfloor$ and $\mathrm{GCD}(t_j, g_j) \neq 1$ reconstructions, see *e.g.* [Monagan '04]
- construct large m with Chinese Remaindering:
  construct solution modulo $m = p_1 \cdots p_N$ from solutions modulo machine-sized primes p_i

A fast rational solver

INPUT: $I_{\mathbb{Q}}$ unreduced rational matrix

OUTPUT: $O_{\mathbb{Q}}$ row reduced rational matrix

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

Function reconstruction

univariate rational function $\mathbb{Q}[d]$ reconstruction:

- works similar to the case $\mathbb{Q}$
- Chinese remaindering becomes Lagrange polynomial interpolation:

$$p_1 \cdots p_N \to (d - p_1) \cdots (d - p_N)$$

- rational reconstruction becomes Pade approximation:

$$\text{interpolating polynomial} \to \text{rational function}$$

multivariate rational function $\mathbb{Q}[d, s, t, \ldots]$ reconstruction:

- by iteration

A fast univariate solver

A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}}[x] & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1\cdot p_2\cdot p_3\cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1\cdot p_2\cdot p_3\cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients

$$
\begin{array}{ccccccccc}
I_{\mathbb{Z}_p[x]} & \xrightarrow[\text{by number } x_i]{\text{sample } x} & I_{\mathbb{Z}_p,x_1} & \xrightarrow[\text{reduction}]{\text{row}} & O_{\mathbb{Z}_p,x_1} & \xrightarrow[\text{interpolation}]{\text{polynomial}} & O_{\mathbb{Z}_p[x]} & \xrightarrow[\text{reconstruction}]{\text{rational function}} & O_{\mathbb{Z}_p[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_p,x_2} & \longrightarrow & O_{\mathbb{Z}_p,x_2} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_p,x3} & \longrightarrow & O_{\mathbb{Z}_p,x3} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

note: massively parallisable

```
 ____  __  __ _  ____  ____  ____
(  __)(  )(  ( \(  _ \(  __)(    \
 ) _)  )( /    / )   / ) _)  ) D (
(__)  (__)\_)__)(__\_)(____)(____/
```

```
Package: finred
Author:  Andreas v. Manteuffel
```

features:

- C++11 implementation for univariate sparse matrices
- employs `flint` library
- parallelisation: SIMD, threads, MPI, batch
- equation filtering: eliminate redundant rows
- plus lots of IBP specific features
- much faster than Reduze 2

Part III: Results for four-loop form factors

[AvM, Schabinger]

Results for massless QCD @ 4 loops

[AvM, Schabinger '16]

completed:

- N_f^3 for quarks and gluons (three massless quark loops)
- complexity: 12 denominators, 6 numerators, non-planar, $O(10^8)$ eqs. per sector
- master integrals: d dimensional solutions via ${}_pF_q$ and Γ functions

checks:

- reductions verified against at least 5 independent samples
- calculation performed in different gauges
  - general R_ξ gauge, general external polarisation vectors
  - background field gauge

  result independent of these choices
- two independent diagram evaluations:
  - Qgraf + Mathematica
  - Qgraf + Form
- poles through to $1/\epsilon^3$ [Moch, Vermaseren, Vogt '05] reproduced

remarks:

- general R_ξ gauge introduces many dots

QCD result @ 4-loops for quarks

[AvM, Schabinger '16]

bare quark form factor

$$\mathcal{F}_4^q|_{N_f^3} = C_F \Bigg[\frac{1}{\epsilon^5}\left(\frac{1}{27}\right) + \frac{1}{\epsilon^4}\left(\frac{11}{27}\right) + \frac{1}{\epsilon^3}\left(\frac{4}{9}\zeta_2 + \frac{254}{81}\right) + \frac{1}{\epsilon^2}\left(-\frac{26}{27}\zeta_3 + \frac{44}{9}\zeta_2 + \frac{29023}{1458}\right) + \frac{1}{\epsilon}\left(\frac{23}{3}\zeta_4 - \frac{286}{27}\zeta_3 + \frac{1016}{27}\zeta_2 + \frac{331889}{2916}\right) - \frac{146}{9}\zeta_5 - \frac{104}{9}\zeta_2\zeta_3 + \frac{253}{3}\zeta_4 - \frac{6604}{81}\zeta_3 + \frac{58046}{243}\zeta_2 + \frac{10739263}{17496} + \mathcal{O}(\epsilon) \Bigg]$$

cusp anomalous dimension:

$$\Gamma_4^q|_{N_f^3} = C_F \left[\frac{64}{27}\zeta_3 - \frac{32}{81}\right]$$

agrees with [Grozin, Henn, Korchemsky, Marquard '15], [Henn, Smirnov, Smirnov, Steinhauser '16]

First QCD result @ 4-loops for gluons

[AvM, Schabinger '16]

bare gluon form factor

$$
\begin{aligned}
\mathcal{F}_4^g|_{N_f^3} = C_F \Bigg[& -\frac{2}{3\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{32}{3}\zeta_3 - \frac{145}{9}\right) + \frac{1}{\epsilon}\left(\frac{352}{45}\zeta_2^2 + \frac{1040}{9}\zeta_3 + \frac{68}{9}\zeta_2 - \frac{10003}{54}\right) \\
& + \frac{4288}{27}\zeta_5 - 64\zeta_3\zeta_2 + \frac{2288}{27}\zeta_2^2 + \frac{24812}{27}\zeta_3 + \frac{3074}{27}\zeta_2 - \frac{508069}{324} + \mathcal{O}(\epsilon) \Bigg] \\
+ C_A \Bigg[& \frac{1}{27\epsilon^5} + \frac{5}{27\epsilon^4} + \frac{1}{\epsilon^3}\left(-\frac{14}{27}\zeta_2 - \frac{55}{81}\right) + \frac{1}{\epsilon^2}\left(-\frac{586}{81}\zeta_3 - \frac{70}{27}\zeta_2 - \frac{24167}{1458}\right) \\
& + \frac{1}{\epsilon}\left(-\frac{802}{135}\zeta_2^2 - \frac{5450}{81}\zeta_3 - \frac{262}{81}\zeta_2 - \frac{465631}{2916}\right) - \frac{14474}{135}\zeta_5 + \frac{4556}{81}\zeta_3\zeta_2 \\
& - \frac{1418}{27}\zeta_2^2 - \frac{99890}{243}\zeta_3 + \frac{38489}{729}\zeta_2 - \frac{20832641}{17496} + \mathcal{O}(\epsilon) \Bigg]
\end{aligned}
$$

gluon cusp anomalous dimension:

$$
\Gamma_4^g|_{N_f^3} = C_A \left[\frac{64}{27}\zeta_3 - \frac{32}{81}\right]
$$

- respects Casimir scaling
- non-planar C_F pieces do not contribute to $\Gamma_4^g|_{N_f^3}$

Conclusions

basis of finite integrals:

- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations (also see HH, Hj !)

reductions via finite field sampling:

- speeds up integration-by-parts reductions
- useful also in other contexts

four-loop form factors:

- warmup: N_f^3 contributions to quark and gluon form factor
- more to come soon