TWO-PARTON SCATTERING IN THE HIGH-ENERGY LIMIT

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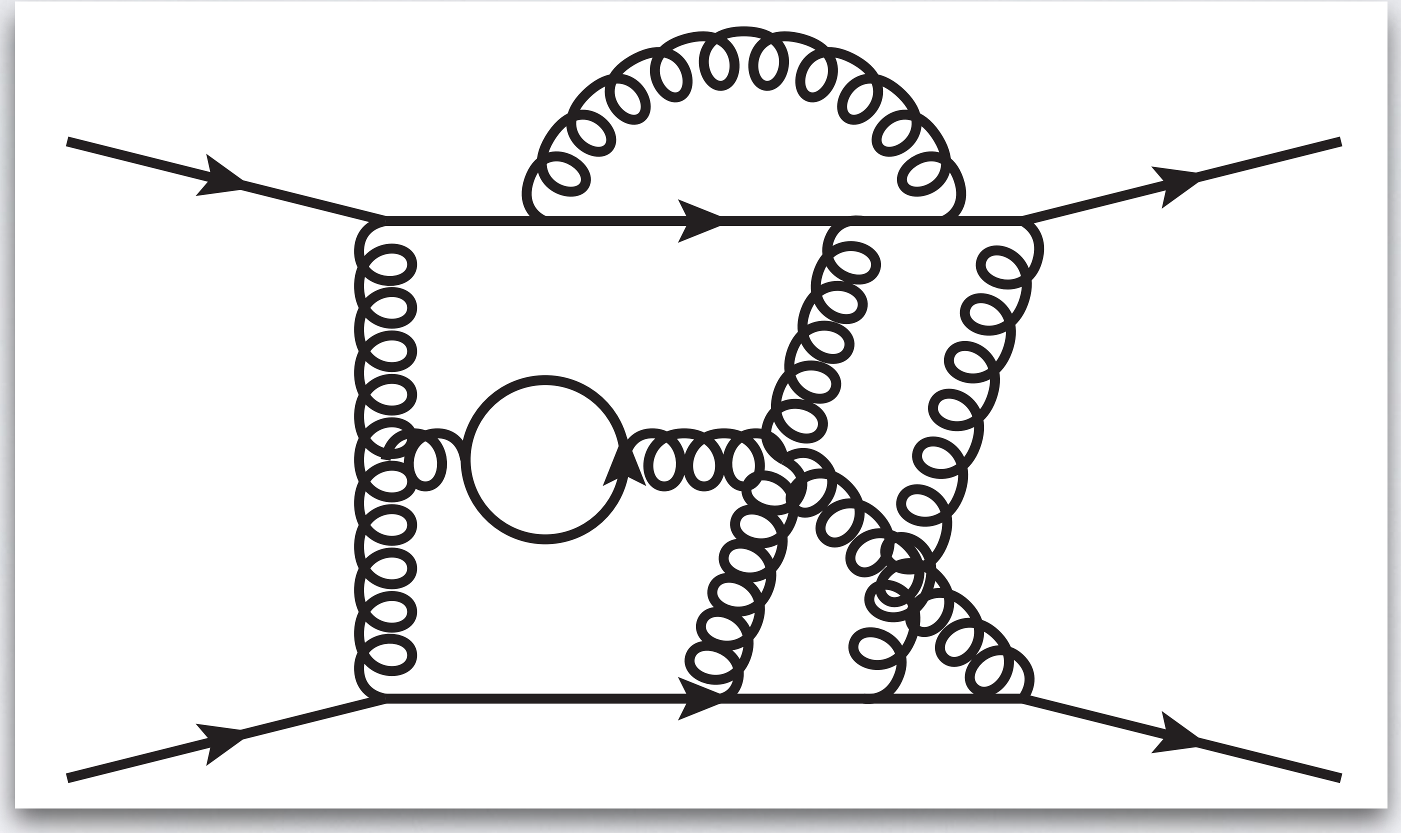
OUTLINE

- **Aspects of 2 → 2 scattering amplitudes in the high-energy limit**
- **High-energy evolution and the Balitsky-JIMWLK equation**
- **The three-Reggeon cut**
- **The two-Reggeon cut**

In collaboration with Simon Caron-Huot, Einan Gardi and Joscha Reichel,

Based on arXiv:1701.05241 and work in progress

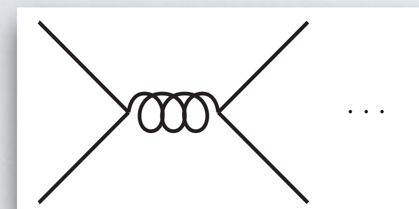
ASPECTS OF 2 → 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

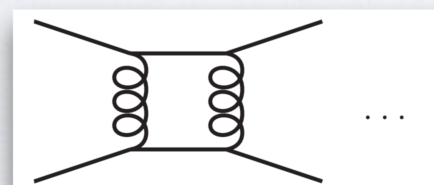


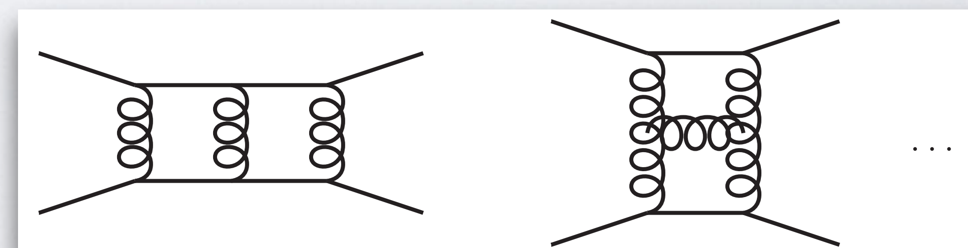
2 ➔ 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

- Calculation of scattering amplitudes at high order in perturbation theory is one of the main ingredients for the program of precision physics at the LHC

$$\mathcal{M} = 4\pi\alpha_s\left[\mathcal{M}^{(0)} + \frac{\alpha_s}{4\pi}\mathcal{M}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2\mathcal{M}^{(2)} + \ldots\right]$$





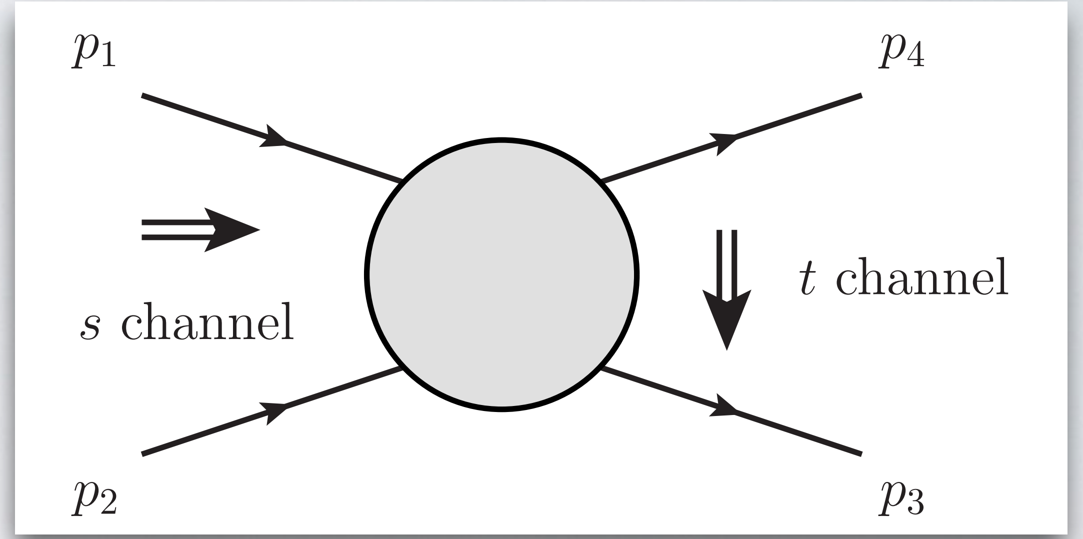


- Amplitudes are complicated functions of the kinematical invariants, their calculation is non-trivial, and it is subject of intense study.

    - Express Feynman integrals in terms of known functions (harmonic polylogarithms, elliptic integrals, etc)
    - Amplitudes contains infrared divergences, which must cancel when summing virtual and real corrections.

$2 \rightarrow 2$ SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

- Information and constraints can be obtained by considering kinematical limits:
    - the number of invariants is reduced;
    - identify factorisation properties and iterative structures of the amplitude;
    - relevant for phenomenology: because of soft and collinear enhancement, differential distributions in specific kinematic limit develops large logarithms, which may spoil the convergence of the perturbative expansion.



- Consider $2 \rightarrow 2$ scattering amplitudes in the high-energy limit:

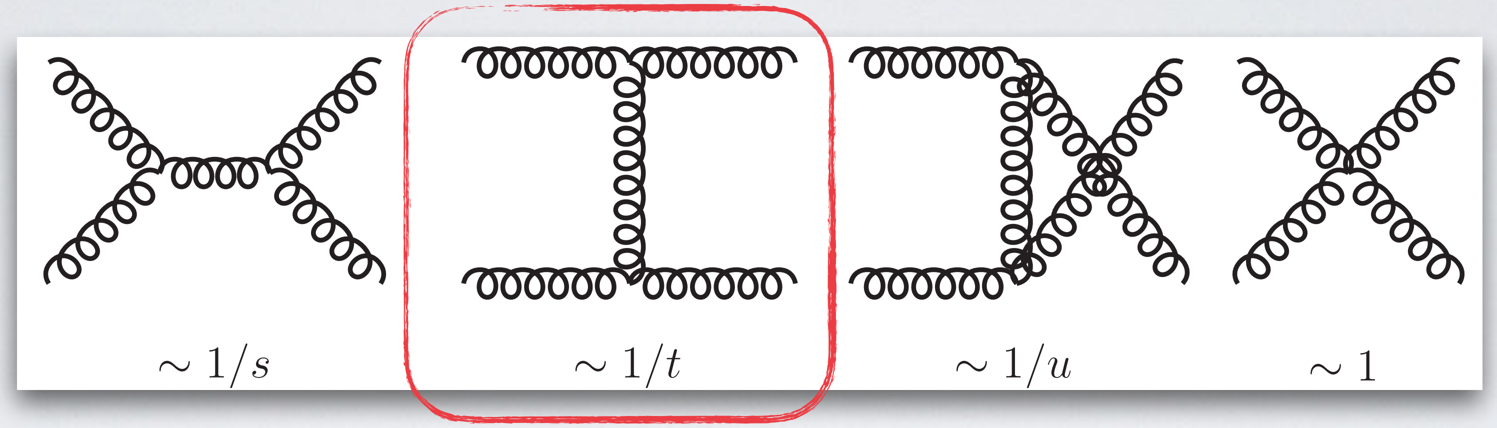
$$s = (p_1 + p_2)^2 \gg -t = -(p_1 - p_4)^2 > 0$$

- The amplitude becomes a function of the ratio $|s/t|$; here we consider the leading power term in this expansion

$$\mathcal{M}(s,t,\mu) = \mathcal{M}_{LP}\left(\frac{s}{-t}, \frac{-t}{\mu^2}\right)\left[1 + \mathcal{O}\left(\frac{-t}{s}\right)\right].$$

2 ➔ 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

- Gluon-gluon scattering amplitude at tree level:



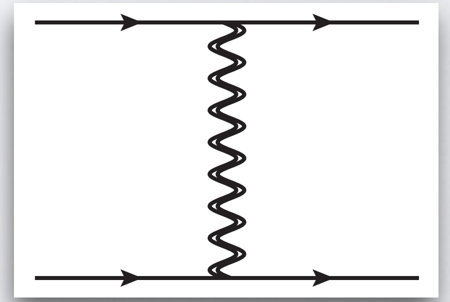
- In the high-energy limit only the second diagram contributes at leading power.

$$\mathcal{M}^{(0)}_{ij\to ij} = \frac{2s}{t}\,(T_i^b)_{a_1a_4}(T_j^b)_{a_2a_3}\,\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}.$$

- The amplitude at higher orders contains logarithms of the ratio |s/t|. They can be characterised in terms of Regge poles and cuts: at LL

Regge, Gribov

$$\mathcal{M}_{ij\to ij}|_{\rm LL} = \left(\frac{s}{-t}\right)^{\frac{\alpha_s}{\pi}\,C_A\,\alpha_g^{(1)}(t)} 4\pi\alpha_s\,\mathcal{M}^{(0)}_{ij\to ij},$$

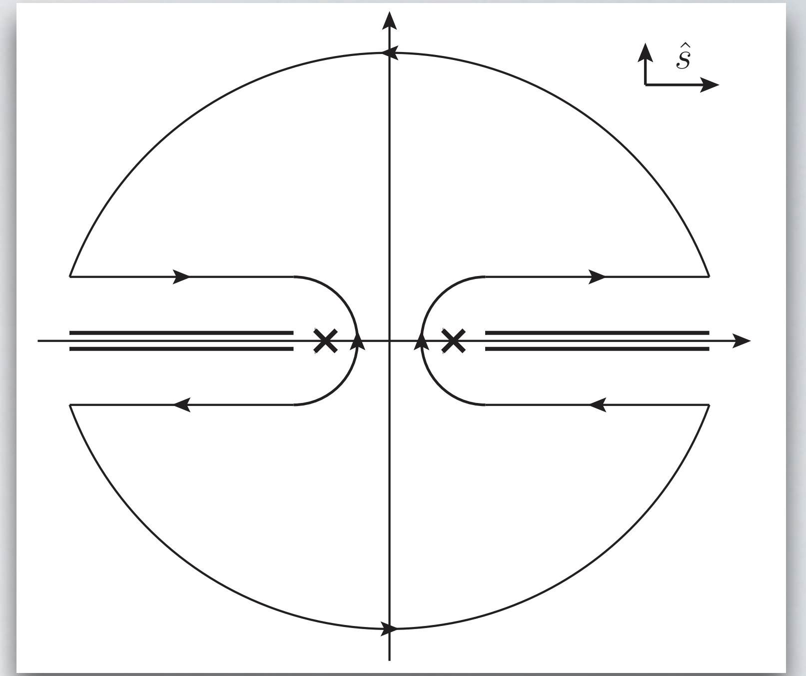


- The function $\alpha_g(t)$ is known as the Regge trajectory

$$\alpha_g^{(1)}(t) = \frac{r_\Gamma}{2\epsilon}\left(\frac{-t}{\mu^2}\right)^{-\epsilon} \overset{\mu^2\to -t}{=} \frac{r_\Gamma}{2\epsilon}, \qquad r_\Gamma = e^{\epsilon\gamma_{\rm E}}\,\frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \approx 1 - \frac{1}{2}\zeta_2\,\epsilon^2 - \frac{7}{3}\zeta_3\,\epsilon^3 + \ldots$$

2 → 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

- Determining the amplitude beyond LL requires to understand the structure of Regge cuts.



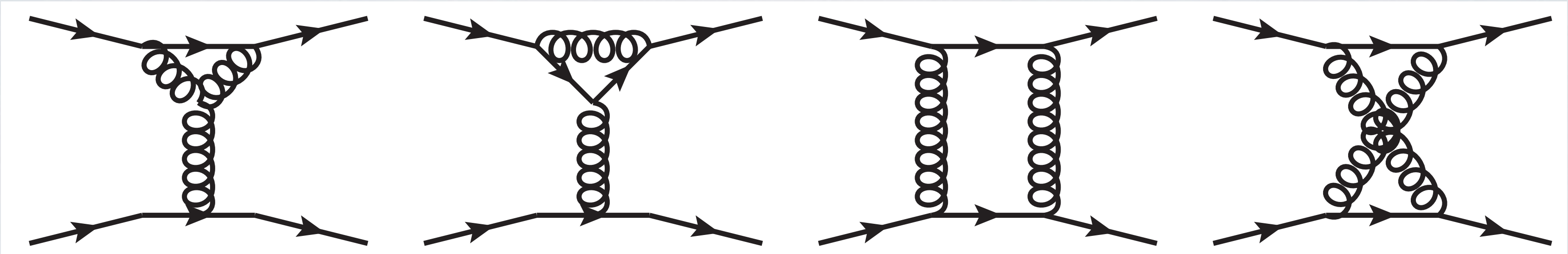
- The amplitudes which develop definite factorisation properties in the high-energy limit are the so called even and odd amplitudes, i.e. the projection onto eigenstates of signature, (crossing symmetry s ↔ u)

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big).$$

- $M^{(+)}$ and $M^{(-)}$ are respectively imaginary and real, when expressed in terms of the natural signature-even combination of logs

$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right).$$

2 ➔ 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT



- Beyond tree level the amplitude has a non-trivial color structure

$$\mathcal{M}(s,t) = \sum_i c^{[i]}\, \mathcal{M}^{[i]}(s,t).$$

- Decompose the amplitude in a color orthonormal basis in the t-channel

$$8 \otimes 8 = 1 \oplus 8_s \oplus 8_a \oplus 10 \oplus \overline{10} \oplus 27 \oplus 0$$

- Invoking Bose symmetry we deduce

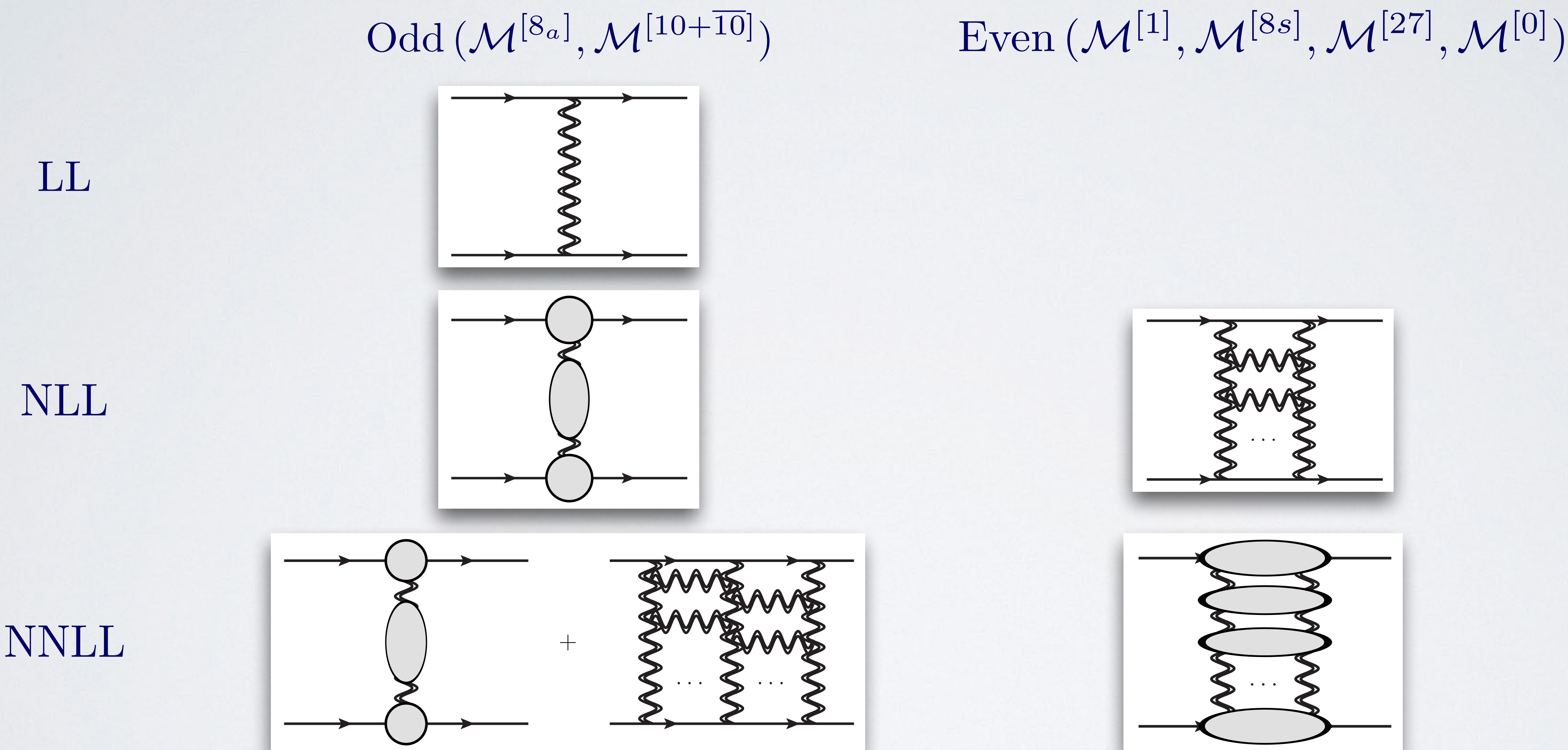
$$\text{odd:}\ \mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]}, \qquad \text{even:}\ \mathcal{M}^{[1]}, \mathcal{M}^{[8_s]}, \mathcal{M}^{[27]}, \mathcal{M}^{[0]} \qquad (gg\ \text{scattering})\,.$$

FACTORISATION STRUCTURE

- Write the amplitude as the sum of odd and even component

$$\mathcal{M}(s,t) = \mathcal{M}^{(-)}(s,t) + \mathcal{M}^{(+)}(s,t), \qquad \mathcal{M}^{(\pm)}(s,t) = 4\pi\alpha_s \sum_{l,m} \left(\frac{\alpha_s}{\pi}\right)^l L^m \, \mathcal{M}^{(\pm,l,m)} \, .$$

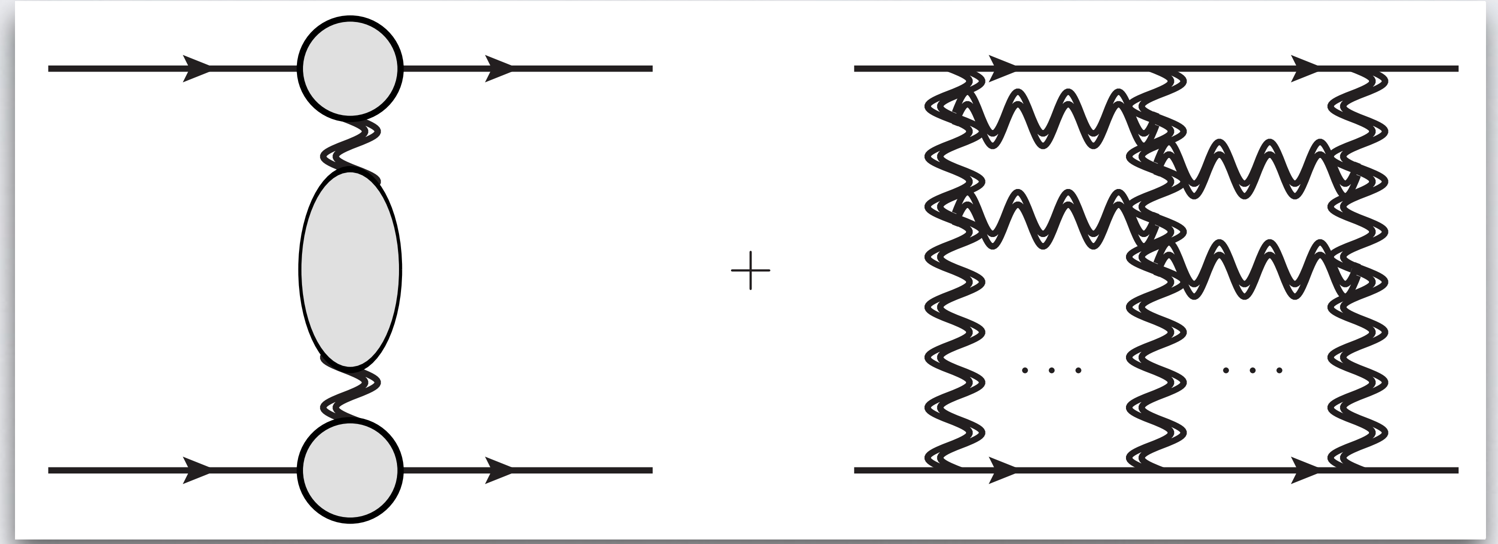
- The amplitude in the high-energy limit has the following factorisation structure



- Focus on the Regge-cut contributions: define a “reduced” amplitude by removing the Reggeized gluon and collinear divergences

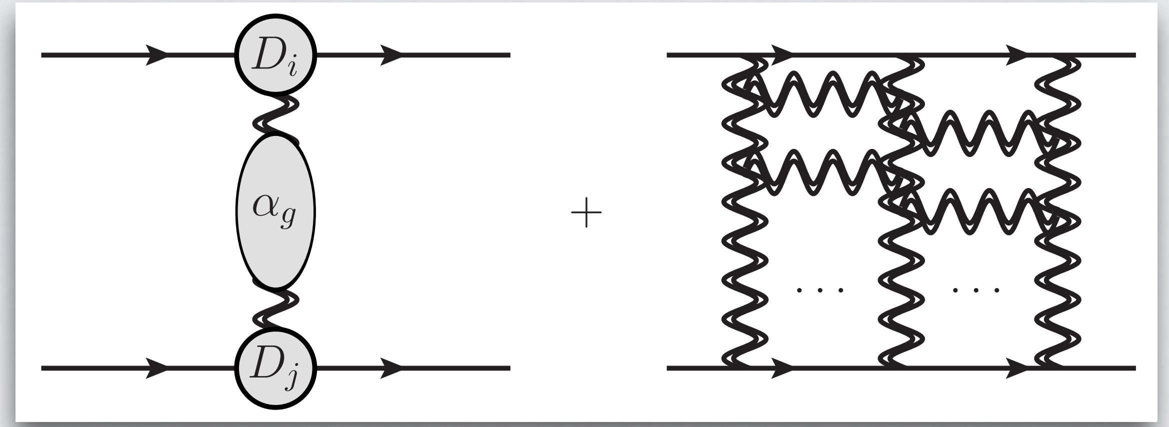
$$\hat{\mathcal{M}}_{ij\to ij} \equiv (Z_i Z_j)^{-1} \, e^{-\mathbf{T}_t^2 \, \alpha_g(t) \, L} \, \mathcal{M}_{ij\to ij} \, ,$$

THE BALITSKY-JIMWLK EQUATION AND THE THREE REGGEON CUT



THE ODD AMPLITUDE AT NNLL

- Starting at NNLL, one has mixing between one- and three-Reggeons exchange:

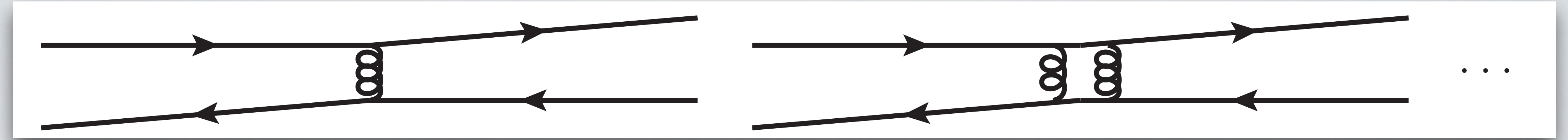


Del Duca, Glover, 2001;
Del Duca, Falcioni, Magnea, LV, 2013

- The mixing between one- and three-Reggeons exchange has significant consequences:
  - It is at the origin of the breaking of the simple power law one has up to NLL accuracy. Such breaking appears for the first time at two loops.
  - Starting at three loops, there will be a single-logarithmic contribution originating from the three-Reggeon exchange, and from the interference of the one- and three-Reggeon exchange: the interpretation of the Regge trajectory at three loops needs to be clarified.
- Schematically, the whole amplitude at NNLL is composed of

$$\hat{\mathcal{M}}_{ij\to ij}|_{\text{NNLL}} = \hat{\mathcal{M}}^{(-)}_{ij\to ij}|_{\text{1-Reggeon + 3-Reggeon}} + \hat{\mathcal{M}}^{(+)}_{ij\to ij}|_{\text{2-Reggeon}}.$$

BFKL THEORY ABRIDGED



- The high-energy limit correspond to a configuration of forward scattering:

$$t = (p_1 - p_4)^2 = (p_2 - p_3)^2 = -\frac{s}{2}(1 - \cos\theta), \qquad s \gg -t \quad \Rightarrow \theta \to 0$$

- The high-energy logarithm is the rapidity difference between the target and the projectile:

$$\eta = L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}.$$

- This kinematical configuration is described in terms of Wilson lines stretching from $-\infty$ to $+\infty$. The Wilson lines follow the paths of color charges inside the projectile, are null and labelled by transverse coordinates **z**:

Korchemskaya, Korchemsky, 1994, 1996

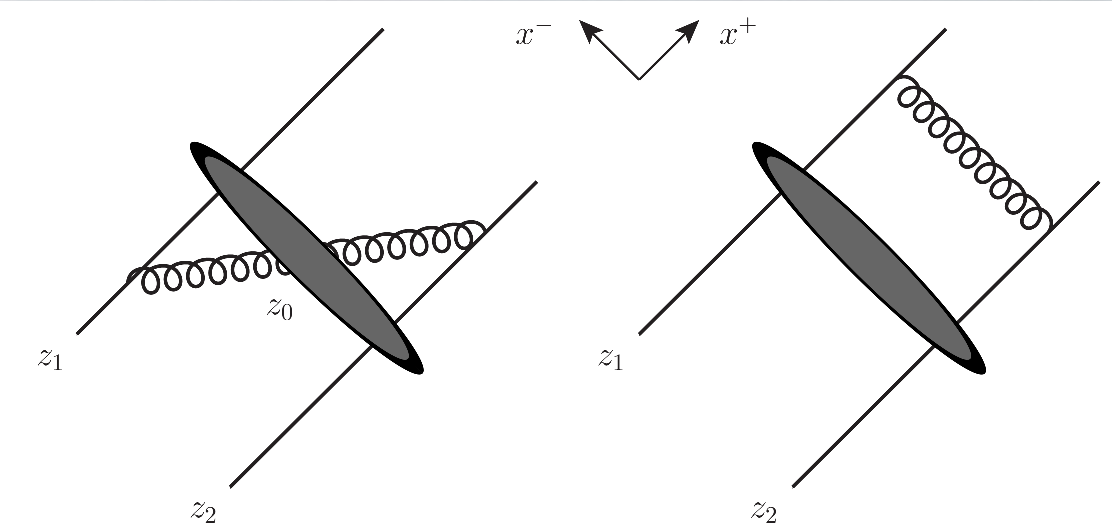
$$U(z_\perp) = \mathcal{P}\exp\left[ig_s \int_{-\infty}^{+\infty} A_+^a(x^+, x^-{=}0, z_\perp)dx^+ T^a\right].$$

- The idea is to approximate, to leading power, the fast projectile and target by Wilson lines and then compute the scattering amplitude between Wilson lines.

Babansky, Balitsky, 2002, Caron-Huot, 2013

THE BALITSKY-JIMWLK EQUATION

- The Wilson line stretches from $-\infty$ to $+\infty$ and thus develops rapidity divergencies. The regularised Wilson lines obeys the Balitsky-JIMWLK evolution equation:



$$-\frac{d}{d\eta}\Big[U(z_1)\dots U(z_n)\Big] = \sum_{i,j=1}^{n} H_{ij}\cdot\Big[U(z_1)\dots U(z_n)\Big],$$

Caron-Huot, 2013

with

$$H_{ij} = \frac{\alpha_s}{2\pi^2}\int [dz_i][dz_j][dz_0]K_{ij;0}\Big[T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\rm ad}(z_0)\left(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R}\right)\Big] + \mathcal{O}(\alpha_s^2).$$

and $T_{L/R}$'s are generators for left and right color rotations:

Balitsky Chirilli, 2013; Kovner, Lublinsky, Mulian, 2013, 2014, 2016

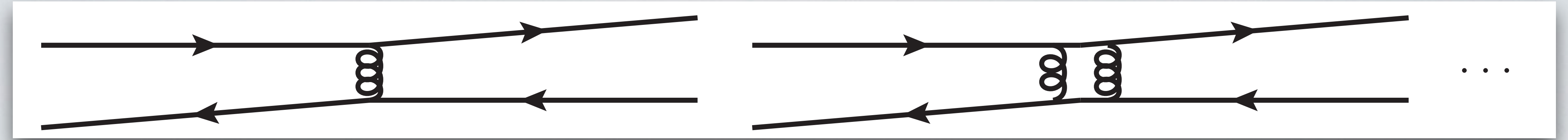
$$T^a_{i,L} = [T^a U(z_i)]\,\frac{\delta}{\delta U(z_i)}, \qquad T^a_{i,R}(z) = [U(z_i)T^a]\,\frac{\delta}{\delta U(z_i)}.$$

- In our analysis we need only the leading-order conformal invariant kernel K_{ij}

$$K_{ij;0} = S_\epsilon(\mu^2)\frac{\Gamma(1-\epsilon)^2}{\pi^{-2\epsilon}}\,\frac{z_{0i}\cdot z_{0j}}{(z_{0i}^2 z_{0j}^2)^{1-\epsilon}}.$$

- The number of Wilson lines is not fixed: a projectile necessarily contains multiple color charges at different transverse positions.

BFKL THEORY ABRIDGED



- However, in perturbation theory the unitary matrices U(z) will be close to identity and so can be usefully parametrised by a field W

$$U(z) = e^{ig_s T^a W^a(z)} \, .$$

Caron-Huot, 2013

- The color-adjoint field W sources a BFKL Reggeised gluon. A generic projectile, created with four-momentum p_1 and absorbed with p_4, can thus be expanded at weak coupling as

$$|\psi\rangle \sim g_s \, D_1(t) \, |W\rangle + g_s^2 \, D_2(t) \, |WW\rangle + g_s^3 \, D_3(t) \, |WWW\rangle + \ldots = |\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle + \ldots .$$

  and we introduce the impact factors $D_{i,j}$, which encode the dependence on the transverse coordinates of the W fields.

- We need to derive the evolution equation for the field W. This is equivalent to switch from the Balitsky-JIMWLK to the BFLK regime.

THE BALITSKY-JIMWLK EQUATION

- Expand U in powers of W

$$U = e^{ig_s\, W^a T^a} = 1 + ig_s\, W^a\, T^a - \frac{g_s^2}{2} W^a W^b\, T^a T^b - i\frac{g_s^3}{6} W^a W^b W^c\, T^a T^b T^c + \frac{g_s^4}{24} W^a W^b W^c W^d\, T^a T^b T^c T^d + \mathcal{O}(g_s^5\, W^5).$$

- The expansion of the color generators follows by using the Backer-Campbell-Hausdorff formula. Then, it is possible to expand the leading Hamiltonian H_{ij} in powers of g_s

$$H = H_{k\to k} + H_{k\to k+2} + \ldots$$

We get

$$H_{k\to k} = \frac{\alpha_s C_A}{2\pi^2} \int [dz_i][dz_0] K_{ii;0}\, (W_i - W_0)^a \frac{\delta}{\delta W_i^a} - \frac{\alpha_s}{2\pi^2} \int [dz_i][dz_j][dz_0] K_{ij;0} (W_i - W_0)^x (W_j - W_0)^y (F^x F^y)^{ab} \frac{\delta^2}{\delta W_i^a \delta W_j^b}.$$

- The first non-linear correction is new

$$H_{k\to k+2} = \frac{\alpha_s^2}{3\pi} \int [dz_i][dz_0]\, K_{ii;0}\, (W_i - W_0)^x W_0^y (W_i - W_0)^z\, \mathrm{Tr}\big[F^x F^y F^z F^a\big] \frac{\delta}{\delta W_i^a} + \frac{\alpha_s^2}{6\pi} \int [dz_i][dz_j][dz_0]\, K_{ij;0}\, (F^x F^y F^z F^t)^{ab} \Big[(W_i - W_0)^x W_0^y W_0^z (W_j - W_0)^t - W_i^x (W_i - W_0)^y W_0^z (W_j - W_0)^t - (W_i - W_0)^x W_0^y (W_j - W_0)^z W_j^t\Big] \frac{\delta^2}{\delta W_i^a \delta W_j^b}.$$

Caron-Huot, Gardi, LV, 2017

BFKL THEORY ABRIDGED

- The inner product is the scattering amplitude of Wilson lines renormalized to equal rapidity.

$$G_{11'} \equiv \langle W_1 | W_{1'} \rangle = i \, \frac{\delta^{a_1 a_1'}}{p_1^2} \, \delta^{(2-2\epsilon)}(p_1 - p_1') + \mathcal{O}(g_s^2).$$

- Multi-Reggeon correlators are obtained by Wick contractions

Caron-Huot, 2013

$$\langle W_1 W_2 | W_{1'} W_{2'} \rangle = G_{11'} G_{22'} + G_{12'} G_{21'} + \mathcal{O}(g_s^2),$$
$$\langle W_1 W_2 W_3 | W_{1'} W_{2'} W_{3'} \rangle = G_{11'} G_{22'} G_{33'} + (5 \text{ permutations}) + \mathcal{O}(g_s^2),$$
$$\ldots$$

- There are also off-diagonal elements, which can be defined to have zero overlap (at equal rapidity)

$$\langle W_1 W_2 W_3 | W_4 \rangle = \langle W_4 | W_1 W_2 W_3 \rangle = 0.$$

- Choosing the 1-W and 3-W states to be orthogonal, combined with symmetry of the Hamiltonian, (boost invariance)

$$\frac{d}{d\eta} \langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = 0 \quad \Leftrightarrow \quad \langle H \mathcal{O}_1 | \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 | H \mathcal{O}_2 \rangle \equiv \langle \mathcal{O}_1 | H | \mathcal{O}_2 \rangle,$$

- implies that in this scheme $H_{k \to k+2} = H_{k+2 \to k}$. This relation is known as projectile-target duality.

THE BALITSKY-JIMWLK EQUATION

- An m→m+k transition from the leading-order Balitsky-JIMWLK equation is proportional to g_s^{2l+k}. Thus for $k \geq 0$, all the interactions can be extracted from the leading-order equation.

$$H \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix} = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & 0 & H_{5\to1} & \dots \\ 0 & H_{2\to2} & 0 & H_{4\to2} & 0 & \dots \\ H_{1\to3} & 0 & H_{3\to3} & 0 & H_{5\to3} & \dots \\ 0 & H_{2\to4} & 0 & H_{4\to4} & 0 & \dots \\ H_{1\to5} & 0 & H_{3\to5} & 0 & H_{5\to5} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix}$$

$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & 0 & g_s^6 & \dots \\ 0 & g_s^2 & 0 & g_s^4 & 0 & \dots \\ g_s^4 & 0 & g_s^2 & 0 & g_s^4 & \dots \\ 0 & g_s^4 & 0 & g_s^2 & 0 & \dots \\ g_s^6 & 0 & g_s^4 & 0 & g_s^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix}.$$

LO BFKL kernel

From LO B-JIMWLK

Terms in NNLO B-JIMWLK - predicted by symmetry $H = H^T$

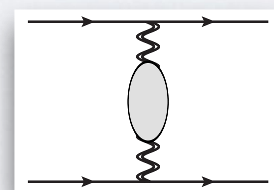
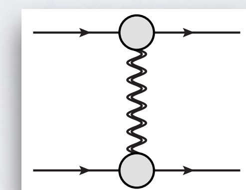
- Interactions with $k < 0$ are suppressed by at least $g_s^{2l+|k|}$, which means that they can first appear in the (|k|+1)-loop Balitsky-JIMWLK Hamiltonian.
- Thus to obtain the m→m−2 transition by direct calculation of the Hamiltonian would require three- loop non-planar computation.
- For our purposes this is unnecessary, since the symmetry of H predicts the result.

THE ODD AMPLITUDE UP TO THREE LOOPS

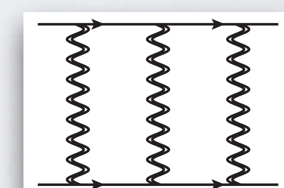
- Ingredients which build up the amplitude: since the odd and even sectors are orthogonal and closed under the action of Ĥ (signature symmetry), we have

$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} \xrightarrow{\text{Regge}} \frac{i}{2s}\left(\hat{\mathcal{M}}^{(+)}_{ij\to ij} + \hat{\mathcal{M}}^{(-)}_{ij\to ij}\right) \equiv \langle\psi_j^{(+)}|e^{-\hat{H}L}|\psi_i^{(+)}\rangle + \langle\psi_j^{(-)}|e^{-\hat{H}L}|\psi_i^{(-)}\rangle.$$

- The signature odd amplitude becomes to three loops:

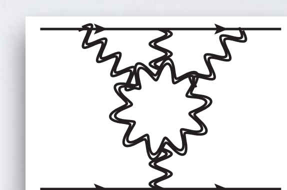



$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{tree}}_{ij\to ij} = \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{LO})},$$

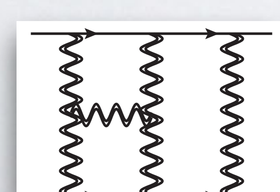


$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{1-loop}}_{ij\to ij} = -L\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{LO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{NLO})},$$

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{2-loops}}_{ij\to ij} = +\frac{1}{2}L^2\langle\psi_{j,1}|(\hat{H}_{1\to1})^2|\psi_{i,1}\rangle^{(\text{LO})} - L\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{NLO})} + \langle\psi_{j,3}|\psi_{i,3}\rangle^{(\text{LO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{NNLO})},$$



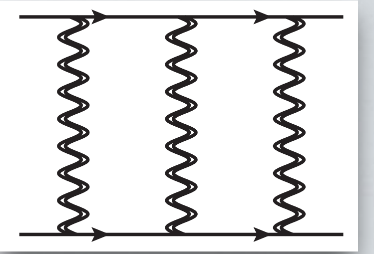
$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{3-loops}}_{ij\to ij} = -\frac{1}{6}L^3\langle\psi_{j,1}|(\hat{H}_{1\to1})^3|\psi_{i,1}\rangle^{(\text{LO})} + \frac{1}{2}L^2\langle\psi_{j,1}|(\hat{H}_{1\to1})^2|\psi_{i,1}\rangle^{(\text{NLO})} - L\Big\{\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{NNLO})} + \Big[\langle\psi_{j,3}|\hat{H}_{3\to3}|\psi_{i,3}\rangle + \langle\psi_{j,3}|\hat{H}_{1\to3}|\psi_{i,1}\rangle + \langle\psi_{j,1}|\hat{H}_{3\to1}|\psi_{i,3}\rangle\Big]^{(\text{LO})}\Big\} + \langle\psi_{j,3}|\psi_{i,3}\rangle^{(\text{NLO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{N}^3\text{LO})}.$$



RESULT: THE ODD AMPLITUDE AT NNLL TO THREE LOOPS

Up to two loops the amplitude reads

Three-Reggeon cut



$$\hat{\mathcal{M}}^{(-,1)}_{ij\to ij} = \left(D_i^{(1)} + D_j^{(1)}\right)\hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

$$\hat{\mathcal{M}}^{(-,2)}_{ij\to ij} = \left[D_i^{(2)} + D_j^{(2)} + D_i^{(1)}D_j^{(1)} + \pi^2 R^{(2)}\left((\mathbf{T}^2_{s-u})^2 - \tfrac{1}{12}(C_A)^2\right)\right]\hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

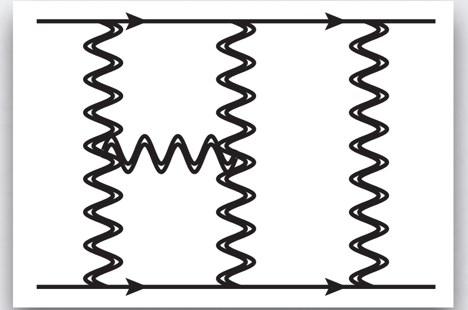
with

$$R^{(2)} \equiv -\frac{1}{24}(r_\Gamma)^2\,\mathcal{I}[1] = -\frac{(r_\Gamma)^2}{6\epsilon^2}\frac{B_{1,1+\epsilon}(\epsilon)}{B_{1,1}(\epsilon)} = (r_\Gamma)^2\left(-\frac{1}{8\epsilon^2} + \frac{3}{4}\epsilon\zeta_3 + \frac{9}{8}\epsilon^2\zeta_4 + \ldots\right),$$

At three loops we find the following amplitude:

$$\hat{\mathcal{M}}^{(-,3,1)}_{ij\to ij} = \pi^2\left(R_A^{(3)}\,\mathbf{T}^2_{s-u}[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}] + R_B^{(3)}\,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]\mathbf{T}^2_{s-u} + R_C^{(3)}\,(C_A)^3\right)\hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

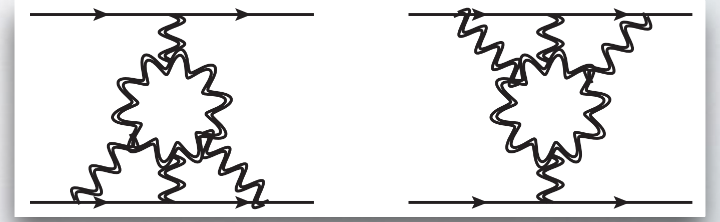
where the loop functions $R_{A,B,C}$ are



$$R_A^{(3)} = \frac{1}{16}(r_\Gamma)^3(\mathcal{I}_a - \mathcal{I}_c) = (r_\Gamma)^3\left(\frac{1}{48\epsilon^3} + \frac{37}{24}\zeta_3 + \ldots\right),$$

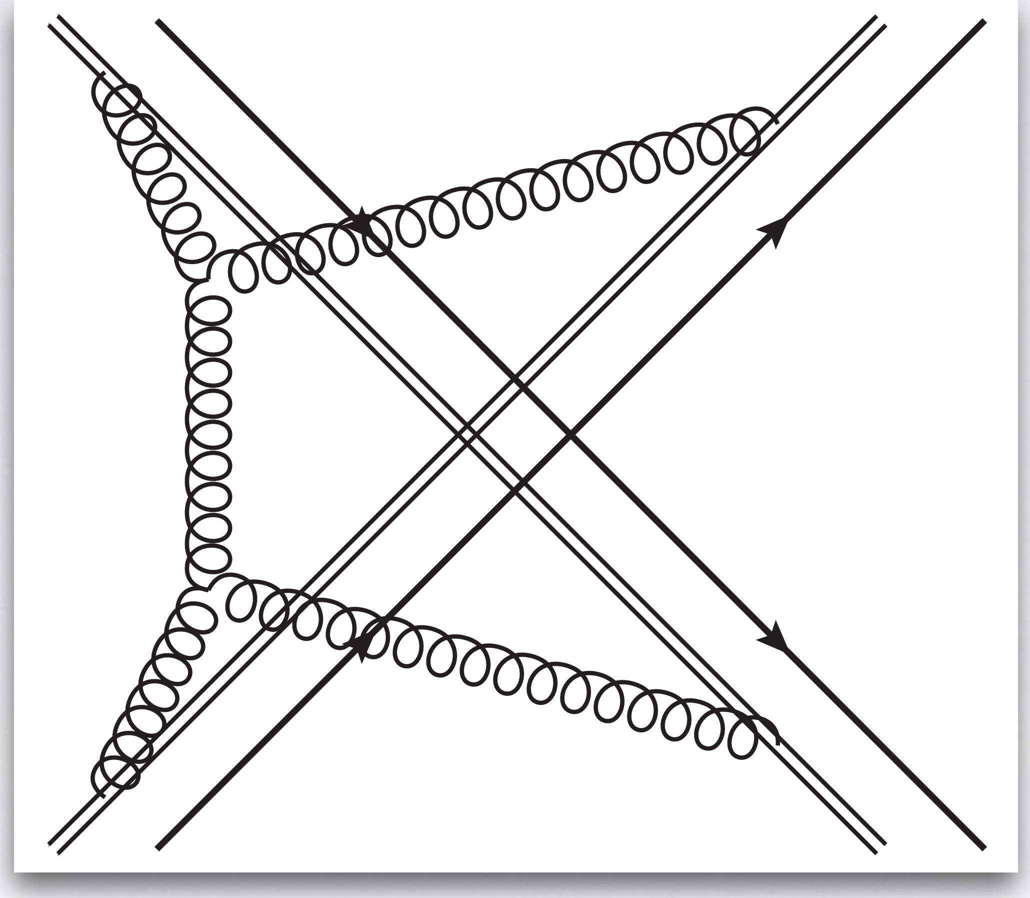
$$R_B^{(3)} = \frac{1}{16}(r_\Gamma)^3(\mathcal{I}_c - \mathcal{I}_b) = (r_\Gamma)^3\left(\frac{1}{24\epsilon^3} + \frac{1}{12}\zeta_3 + \ldots\right),$$

$$R_C^{(3)} = \frac{1}{288}(r_\Gamma)^3(2\mathcal{I}_c - \mathcal{I}_a - \mathcal{I}_b) = (r_\Gamma)^3\left(\frac{1}{864\epsilon^3} - \frac{35}{432}\zeta_3 + \ldots\right).$$



Caron-Huot, Gardi, LV, 2017

COMPARISON BETWEEN REGGE AND INFRARED FACTORIZATION



BFKL VS INFRARED FACTORISATION

- The calculation of the amplitude was based solely on evolution equations of the Regge limit.
- Highly nontrivial consistency test: the prediction must be consistent with the known exponentiation pattern and the anomalous dimensions governing infrared divergences.
- Conversely, the prediction for the reduced amplitude gives a constraint on the soft anomalous dimension.
- The infrared divergences of amplitudes are controlled by a renormalization group equation:

Becher, Neubert, 2009; Gardi, Magnea, 2009

$$\mathcal{M}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

where $\mathbf{Z}$ is given as a path-ordered exponential of the soft-anomalous dimension:

$$\mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\}.$$

- The soft anomalous dimension for scattering of massless partons ($p_i^2 = 0$) is an operators in color space given, to three loops, by

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

Becher, Neubert, 2009; Dixon, Gardi, Magnea, 2009; Del Duca, Duhr, Gardi, Magnea, White, 2011; Neubert, LV, 2012, Almelid, Duhr, Gardi, McLeod, White, 2017

BFKL VS INFRARED FACTORISATION

- $\boldsymbol{\Gamma}^{\text{dip}}_n$ involves only pairwise interactions amongst the hard partons: "dipole formula"

$$\mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = -\frac{\gamma_K(\alpha_s)}{2}\sum_{i<j}\log\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_i \gamma_i(\alpha_s)\,.$$

- The term $\boldsymbol{\Delta}_n(\rho_{ijkl})$ involves interactions of up to four partons: "quadrupole correction"

$$\mathbf{\Delta}_n(\{\rho_{ijkl}\}) = \sum_{i=3}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^i \mathbf{\Delta}_n^{(i)}(\{\rho_{ijkl}\}).$$

- The three loop correction has been calculated recently, and reads

$$\begin{aligned}\mathbf{\Delta}_n^{(3)}(\{\rho_{ijkl}\}) = \frac{1}{4} f^{abe} f^{cde} \sum_{1\le i<j<k<l\le n} \Big[&\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ikjl},\rho_{iljk}) \\ &+ \mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ijkl},\rho_{ilkj}) + \mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\,\mathcal{F}(\rho_{ijlk},\rho_{iklj})\Big] \\ &- \frac{C}{4} f^{abe} f^{cde} \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n,\\ j,k\ne i}} \{\mathbf{T}_i^a,\mathbf{T}_i^d\}\mathbf{T}_j^b\mathbf{T}_k^c,\end{aligned}$$

Almelid, Duhr, Gardi, 2015, 2016

where F is a function of cross ratios: $\rho_{ijkl} = (-s_{ij})(-s_{kl})/(-s_{ik})(-s_{jl})$. Explicitly, one has

$$\mathcal{F}(\rho_{ikjl},\rho_{ilkj}) = F(1-z_{ijkl}) - F(z_{ijkl}), \quad \text{with} \quad F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2\Big(\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)\Big),$$

where the $\mathscr{L}$ are Brown's single-valued harmonic polylogarithms, and the constant term reads

$$C = \zeta_5 + 2\zeta_2\zeta_3.$$

BFKL VS INFRARED FACTORISATION

Del Duca, Duhr, Gardi, Magnea, White, 2011

- In the high-energy limit the dipole formula reduces to

$$\Gamma^{\rm dip.}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) \xrightarrow{\rm Regge} \frac{\gamma_K(\alpha_s)}{2}\left[L\,\mathbf{T}_t^2 + i\pi\,\mathbf{T}_{s-u}^2 + \frac{C_{\rm tot}}{2}\log\frac{-t}{\lambda^2}\right] + \sum_{i=1}^{4}\gamma_i(\alpha_s) + \mathcal{O}\left(\frac{t}{s}\right),$$

- The quadrupole correction has only one imaginary term at NNLL

Caron-Huot, Gardi, LV, 2017

$$\mathbf{\Delta}^{(3)} \;=\; i\pi\,[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]\,\frac{\zeta_3}{4}\,L + \mathcal{O}(L^0).$$

- Because of the form of $\Gamma^{\rm dip}$ and $\Delta(\rho_{ijkl})$ in the High-energy limit, the $\mathbf{Z}$ factor factorises

$$\mathbf{Z}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \tilde{\mathbf{Z}}\left(\frac{s}{t},\mu,\alpha_s(\mu^2)\right) Z_i\left(t,\mu,\alpha_s(\mu^2)\right) Z_j\left(t,\mu,\alpha_s(\mu^2)\right),$$

- The relevant bit for us is

$$\tilde{\mathbf{Z}}\left(\frac{s}{t},\mu,\alpha_s(\mu^2)\right) = \exp\left\{K\left(\alpha_s(\mu^2)\right)\left[L\,\mathbf{T}_t^2 + i\pi\,\mathbf{T}_{s-u}^2\right] + Q_{\mathbf{\Delta}}^{(3)}\right\}$$

- The factors K and Q_Δ involve integrals over the scale

$$K = -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K\left(\alpha_s(\lambda^2)\right) \;=\; \frac{1}{2\epsilon}\,\frac{\alpha_s(\mu^2)}{\pi} + \ldots,$$

$$Q_{\mathbf{\Delta}}^{(3)} = -\frac{\mathbf{\Delta}^{(3)}}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^3 = \frac{\mathbf{\Delta}^{(3)}}{6\epsilon}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^3.$$

BFKL VS INFRARED FACTORISATION

- The finite reminder of the amplitude, i.e. the hard function reads

$$\mathcal{H}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \exp^{-1}\left\{K\left(\alpha_s(\mu^2)\right)\left[L\,\mathbf{T}_t^2 + i\pi\,\mathbf{T}_{s-u}^2\right] + Q_{\boldsymbol{\Delta}}^{(3)}\right\} \cdot \exp\left\{\alpha_g(t) L\,\mathbf{T}_t^2\right\} \hat{\mathcal{M}}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right).$$

- This equation allows us to pass from directly from the reduced amplitude predicted using BFKL theory, to the hard function.
- In particular, the statement that the left-hand-side H is finite, which is equivalent to the exponentiation of infrared divergences, is a highly nontrivial constraint on our result.
- By using Baker-Campbell-Hausdorff formula we get the hard function at each order in perturbation theory. For instance

$$\mathrm{Re}[\mathcal{H}^{(2,0)}] = \Big[D_i^{(2)} + D_j^{(2)} + D_i^{(1)} D_j^{(1)} - \pi^2 R^{(2)} \tfrac{1}{12}(C_A)^2 + \pi^2\left(R^{(2)} + \tfrac{1}{2}(K^{(1)})^2 + K^{(1)}\hat{\alpha}_g^{(1)}\right)(\mathbf{T}_{s-u}^2)^2\Big]\hat{\mathcal{M}}^{(0)}.$$

Del Duca, Falcioni, Magnea, LV, 2013

BFKL VS INFRARED FACTORISATION

- Some coefficients, like the impact factors, are not predicted explicitly from Regge theory.
- The BFLK approach developed here allows us to extract these quantities consistently, and use them to predict higher orders.
- The impact factors at two loops are extracted by taking the projection of the amplitude onto the antisymmetric octet component:

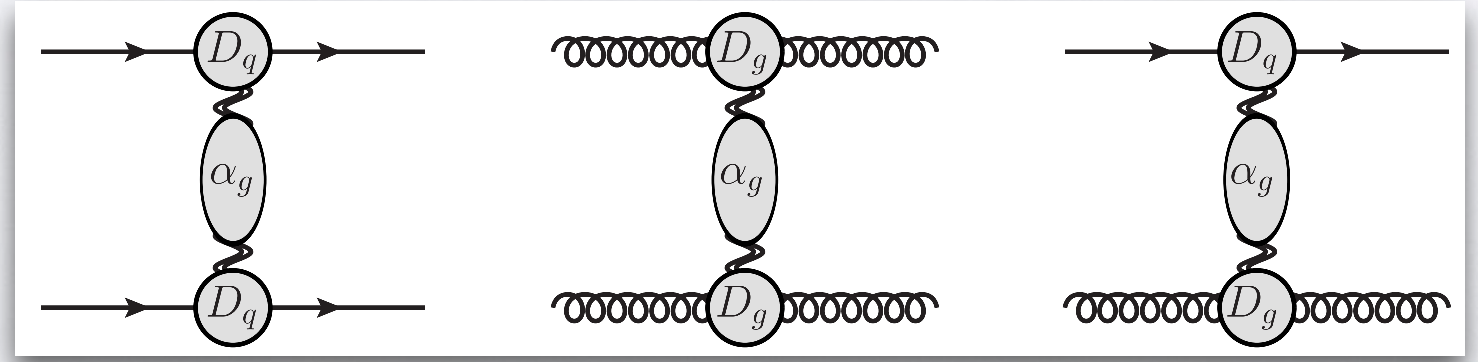
$$2D_g^{(2)} = \frac{\mathcal{H}_{gg\to gg}^{(2,0)[8_a]}}{\mathcal{H}_{gg\to gg}^{(0)[8_a]}} - (D_g^{(1)})^2 + \pi^2 R^{(2)} \frac{N_c^2}{12} - \pi^2 \hat{R}^{(2)} \frac{N_c^2+24}{4},$$

$$D_q^{(2)} + D_g^{(2)} = \frac{\mathcal{H}_{qg\to qg}^{(2,0)[8_a]}}{\mathcal{H}_{qg\to qg}^{(0)[8_a]}} - D_q^{(1)} D_g^{(1)} + \pi^2 R^{(2)} \frac{N_c^2}{12} - \pi^2 \hat{R}^{(2)} \frac{N_c^2+4}{4},$$

$$2D_q^{(2)} = \frac{\mathrm{Re}[\mathcal{H}_{qq\to qq}^{(2,0)[8_a]}]}{\mathcal{H}_{qq\to qq}^{(0)[8_a]}} - (D_q^{(1)})^2 + \pi^2 R^{(2)} \frac{N_c^2}{12} - \pi^2 \hat{R}^{(2)} \frac{N_c^4 - 4N_c^2 + 12}{4N_c^2}.$$

Caron-Huot, Gardi, LV, 2017

- The effect of the three-Reggeon cut is evident from the color-dependent term. Consistency requires the three equations above to be satisfied simultaneously.



BFKL VS INFRARED FACTORISATION

- At three loops, at NNLL, the calculation of the odd sector within Regge theory gives

$$\begin{aligned}
\mathrm{Re}[\mathcal{H}^{(3,1)}] = &\Big[\hat{\alpha}_g^{(3)} + \hat{\alpha}_g^{(2)}\Big(D_i^{(1)} + D_j^{(1)}\Big) + \hat{\alpha}_g^{(1)}\Big(D_i^{(2)} + D_j^{(2)} + D_i^{(1)}D_j^{(1)}\Big)\Big]\mathbf{T}_t^2\,\hat{\mathcal{M}}^{(0)} \\
&+ \pi^2\Big[R_C^{(3)} - \tfrac{1}{12}\hat{\alpha}_g^{(1)}R^{(2)}\Big](\mathbf{T}_t^2)^3\,\hat{\mathcal{M}}^{(0)} + \pi^2\,\hat{\alpha}_g^{(1)}\,\hat{R}^{(2)}\,\mathbf{T}_t^2(\mathbf{T}_{s-u}^2)^2\,\hat{\mathcal{M}}^{(0)} \\
&+ \pi^2\Big[R_A^{(3)} + \tfrac{1}{6}K^{(1)}\Big(2(K^{(1)})^2 + 3\hat{\alpha}_g^{(1)}K^{(1)} + 3\mathbb{d}_2\Big)\Big]\mathbf{T}_{s-u}^2[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\,\hat{\mathcal{M}}^{(0)} \\
&+ \pi^2\Big[R_B^{(3)} - \tfrac{1}{3}K^{(1)}\Big((K^{(1)})^2 + 3\hat{\alpha}_g^{(1)}K^{(1)} + 3(\hat{\alpha}_g^{(1)})^2\Big)\Big][\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\mathbf{T}_{s-u}^2\,\hat{\mathcal{M}}^{(0)}.
\end{aligned}$$

must be finite

which is consistent with infrared factorisation. This is a rather non-trivial check, given that the two calculations are done in two completely different ways.

Caron-Huot, Gardi, LV, 2017

$$\begin{aligned}
\mathrm{Re}[\mathcal{H}^{(3,1)}] = &\Bigg[\hat{\alpha}_g^{(3)} + \hat{\alpha}_g^{(2)}\Big(D_i^{(1)} + D_j^{(1)}\Big) + \hat{\alpha}_g^{(1)}\Big(D_i^{(2)} + D_j^{(2)} + D_i^{(1)}D_j^{(1)}\Big) \\
&+ C_A^2\frac{\pi^2}{864}\Big(\frac{1}{\epsilon^3} - \frac{15\zeta_2}{4\epsilon} - \frac{175\zeta_3}{2}\Big)\Bigg]C_A\,\hat{\mathcal{M}}^{(0)} \\
&+ \pi^2\frac{5\zeta_3}{12}\mathbf{T}_{s-u}^2[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\,\hat{\mathcal{M}}^{(0)} + \pi^2\frac{\zeta_3}{12}[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\mathbf{T}_{s-u}^2\,\hat{\mathcal{M}}^{(0)} + \mathcal{O}(\epsilon).
\end{aligned}$$

BFKL VS INFRARED FACTORISATION

- We get some parts of the finite amplitude. In the orthonormal basis in the t-channel we have

$$\mathrm{Re}[\mathcal{H}^{(3,1),[8_a]}] = \Big\{ C_A \Big[\hat{\alpha}_g^{(3)} + \hat{\alpha}_g^{(2)} \Big(D_i^{(1)} + D_j^{(1)} \Big) + \hat{\alpha}_g^{(1)} \Big(D_i^{(2)} + D_j^{(2)} + D_i^{(1)} D_j^{(1)} \Big) \Big] + C_A^3 \frac{\pi^2}{864} \Big(\frac{1}{\epsilon^3} - \frac{15\zeta_2}{4\epsilon} - \frac{175\zeta_3}{2} \Big) - C_A \pi^2 \frac{2\zeta_3}{3} + \mathcal{O}(\epsilon) \Big\} \hat{\mathcal{M}}^{(0),[8_a]},$$

$$\mathrm{Re}[\mathcal{H}^{(3,1),[10+\overline{10}]}] = \sqrt{2} C_A \sqrt{C_A^2 - 4} \Big\{ \frac{11\pi^2 \zeta_3}{24} + \mathcal{O}(\epsilon) \Big\} \hat{\mathcal{M}}^{(0),[8_a]}.$$

Caron-Huot, Gardi, LV, 2017

- The antisymmetric octet amplitude cannot be predicted entirely, given the unknown Regge trajectory at three loops. The $10+\overline{10}$ component can be predicted exactly, and it agrees with a recent calculation of the gluon-gluon amplitude in N=4 SYM. Henn, Mistlberger, 2016

- Starting from three loops the "gluon Regge trajectory" is scheme-dependent. We define it to be the $1 \to 1$ matrix element of the Hamiltonian, $\alpha_g(t) = -H_{1\to 1}/C_A$, in the scheme where states corresponding to a different number of Reggeon are orthogonal

$$\log \frac{\mathcal{M}^{[8_a]}_{gg\to gg}}{\mathcal{M}^{(0)[8_a]}_{gg\to gg}} = L \Big\{ -H_{1\to 1}(t) + \Big(\frac{\alpha_s}{\pi}\Big)^3 \pi^2 \Big[N_c \Big(-2R_A^{(3)} + 2R_B^{(3)} \Big) + N_c^3 R_C^{(3)} \Big] \Big\} + \mathcal{O}(L^0, \alpha_s^4),$$

THE REGGE TRAJECTORY AT THREE LOOPS IN N=4 SYM

- Thanks to a recent calculation of the gluon-gluon amplitude in N=4 SYM, in this theory one has

$$\log \frac{\mathcal{M}^{[8_a],\mathcal{N}=4}_{gg\to gg}}{\mathcal{M}^{(0)[8_a]}_{gg\to gg}}\Bigg|_L = N_c\left[\frac{\alpha_s}{\pi}k_1 + \left(\frac{\alpha_s}{\pi}\right)^2 k_2 + \left(\frac{\alpha_s}{\pi}\right)^3 k_3 + \cdots\right],$$

Henn, Mistlberger, 2016

Define the Regge trajectory as

$$-H^{\mathcal{N}=4}_{1\to1} = N_c\left[\frac{\alpha_s}{\pi}\alpha_g^{(1)}|_{\mathcal{N}=4} + \left(\frac{\alpha_s}{\pi}\right)^2 \alpha_g^{(2)}|_{\mathcal{N}=4} + \left(\frac{\alpha_s}{\pi}\right)^3 \alpha_g^{(3)}|_{\mathcal{N}=4} + \cdots\right],$$

Then, matching these two results we get

$$\alpha_g^{(1)}|_{\mathcal{N}=4} = k_1 = \frac{1}{2\epsilon} - \epsilon\frac{\zeta_2}{4} - \epsilon^2\frac{7}{6}\zeta_3 - \epsilon^3\frac{47}{32}\zeta_4 + \epsilon^4\left(\frac{7}{12}\zeta_2\zeta_3 - \frac{31}{10}\zeta_5\right) + \mathcal{O}(\epsilon^5),$$

$$\alpha_g^{(2)}|_{\mathcal{N}=4} = k_2 = N_c\left[-\frac{\zeta_2}{8}\frac{1}{\epsilon} - \frac{\zeta_3}{8} - \epsilon\frac{3}{16}\zeta_4 + \epsilon^2\left(\frac{71}{24}\zeta_2\zeta_3 + \frac{41}{8}\zeta_5\right) + \mathcal{O}(\epsilon^3)\right],$$

Caron-Huot, Gardi, LV, 2017

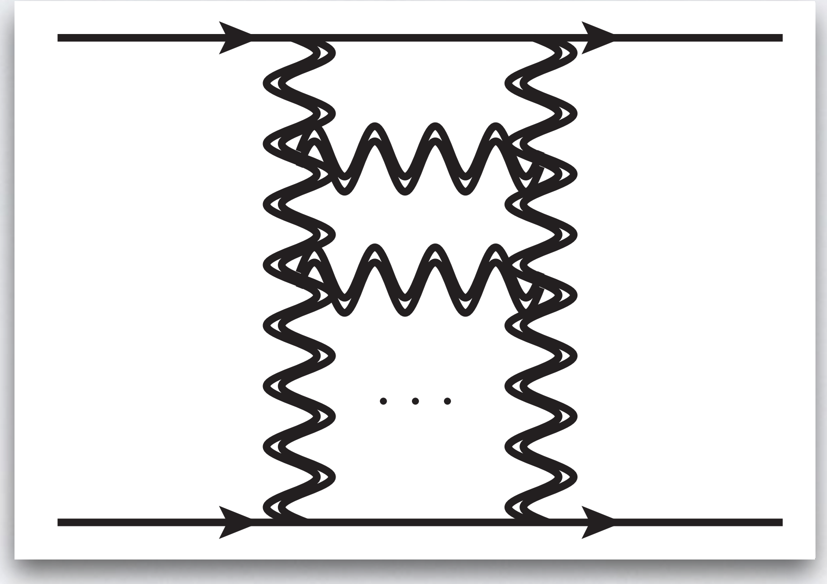
$$\alpha_g^{(3)}|_{\mathcal{N}=4} = k_3 - \pi^2\left[N_c\left(-2R_A^{(3)} + 2R_B^{(3)}\right) + N_c^3 R_C^{(3)}\right]$$

$$= N_c^2\left[-\frac{\zeta_2}{144}\frac{1}{\epsilon^3} + \frac{49\zeta_4}{192}\frac{1}{\epsilon} + \frac{107}{144}\zeta_2\zeta_3 + \frac{\zeta_5}{4} + \mathcal{O}(\epsilon)\right] + N_c^0\left[0 + \mathcal{O}(\epsilon)\right].$$

Del Duca, Falcioni, Magnea, LV, 2014

- The amplitude is really a sum of multiple powers. Simply exponentiating the log of the full amplitude at three loops predicts an incorrect four-loop amplitude. The correct, procedure is to exponentiate the BFKL Hamiltonian. With the "trajectory" fixed as above, this procedure does not require any new parameter for the odd amplitude at NNLL to all loop orders.

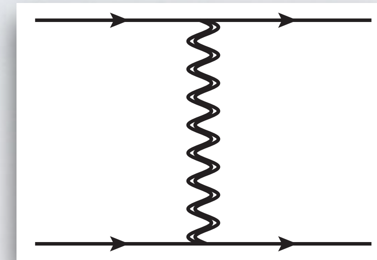
THE BALITSKY-JIMWLK EQUATION AND THE TWO REGGEON CUT



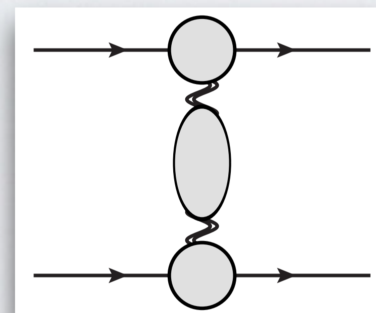
THE BALITSKY-JIMWLK EQUATION AND THE TWO REGGEON CUT

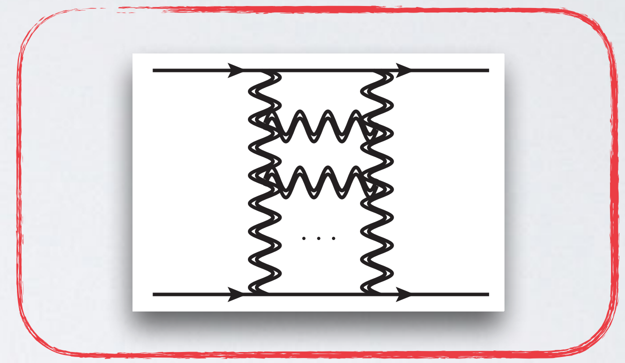
Odd $(\mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]})$ | Even $(\mathcal{M}^{[1]}, \mathcal{M}^{[8s]}, \mathcal{M}^{[27]}, \mathcal{M}^{[0]})$

LL

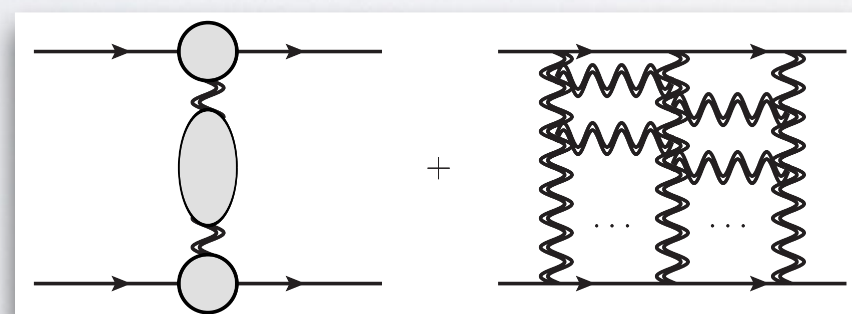


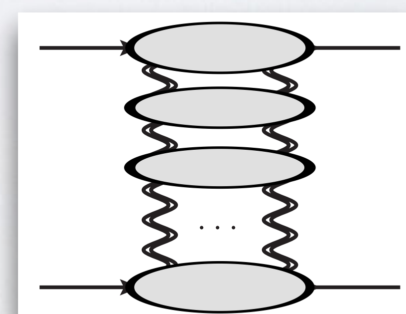
NLL





NNLL

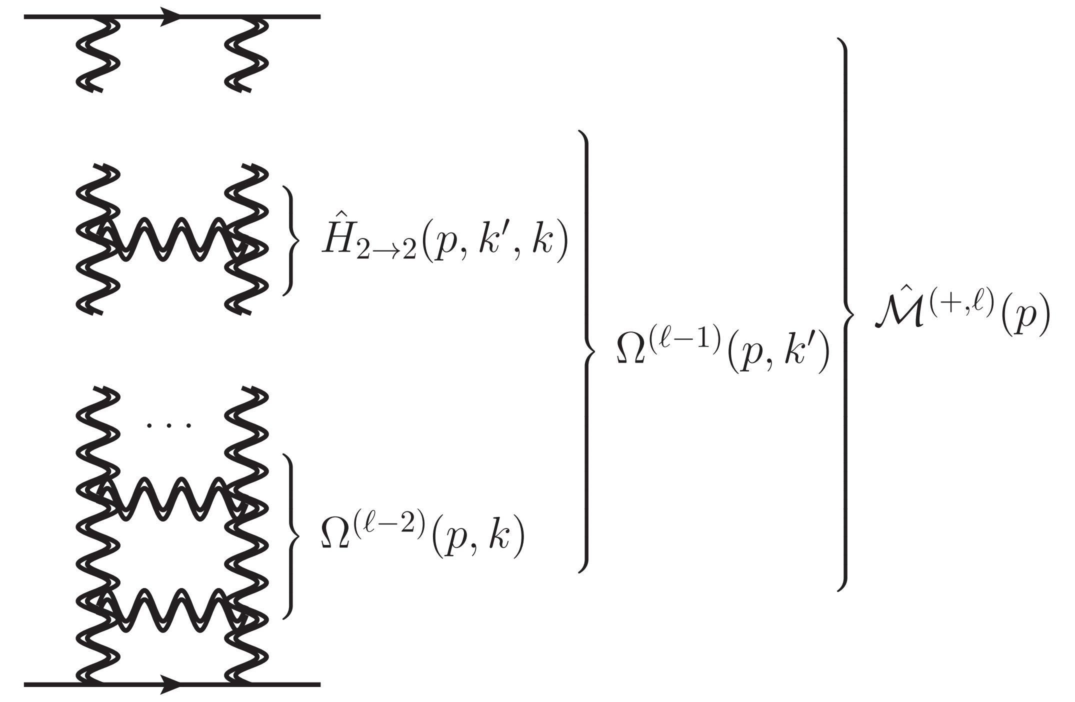




- The even amplitude at NLL is given by

$$\frac{i}{2s}\hat{\mathcal{M}}^{(+)}_{\rm NLL} = \langle \psi^{(+)}_{j,2} | e^{-\hat{H}L} | \psi^{(+)}_{i,2} \rangle^{(\rm LO)}, \qquad \frac{i}{2s}\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = \frac{1}{(\ell-1)!}\langle \psi^{(+)}_{2} | \left(-\hat{H}_{2\to 2}\right)^{\ell-1} | \psi^{(+)}_{2} \rangle^{(\rm LO)}.$$

THE BALITSKY-JIMWLK EQUATION AND THE TWO REGGEON CUT



- The even amplitude reads

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!}\int [{\rm D}k]\,\frac{p^2}{k^2(k-p)^2}\,\Omega^{(\ell-1)}(p,k)\,\mathbf{T}^2_{s-u}\,\mathcal{M}^{(0)},$$

with

$$B_0 = r_\Gamma = e^{\epsilon\gamma_{\rm E}}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}.$$

- The "target averaged wave function" reads

$$\Omega^{(\ell-1)}(p,k) = (2C_A-\mathbf{T}_t^2)\,\Psi^{(\ell-1)}(p,k) + (C_A-\mathbf{T}_t^2)\,\Phi^{(\ell-1)}(p,k),$$

with

$$\Psi^{(\ell-1)}(p,k) = \int [{\rm D}k']\,f(p,k,k')\left[\Omega^{(\ell-2)}(p,k')-\Omega^{(\ell-2)}(p,k)\right], \qquad \Phi^{(\ell-1)}(p,k) = \frac{1-J(p,k)}{2\epsilon}\Omega^{(\ell-2)}(p,k),$$

and the initial condition is fixed to

$$\Omega^{(0)}(p,k)=1.$$

- The function f is the BFKL kernel

$$f(p,k',k) = \frac{k'^2}{k^2(k-k')^2}+\frac{(p-k')^2}{(p-k)^2(k-k')^2}-\frac{p^2}{k^2(p-k)^2}, \qquad J(p,k) = -2\epsilon\int [{\rm D}k']\,f(p,k,k').$$

THE BALITSKY-JIMWLK EQUATION AND THE TWO REGGEON CUT

- Up to four loops one gets

$$\hat{\mathcal{M}}^{(+,1)}_{\rm NLL} = -i\pi \, \frac{B_0}{2\epsilon} \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)},$$

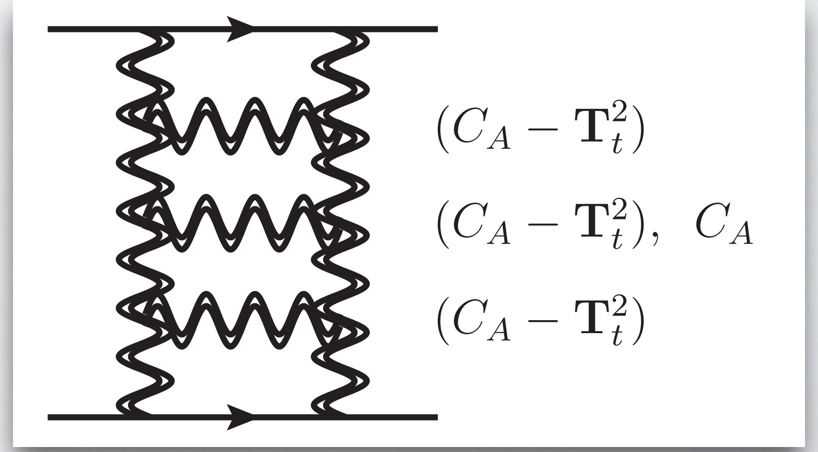
$$\hat{\mathcal{M}}^{(+,2)}_{\rm NLL} = i\pi \, \frac{(B_0)^2}{2} \left[\frac{1}{(2\epsilon)^2} + \frac{9\zeta_3}{2}\epsilon + \frac{27\zeta_4}{4}\epsilon^2 + \frac{63\zeta_5}{2}\epsilon^3 + \mathcal{O}(\epsilon^4) \right] (C_A - \mathbf{T}^2_t) \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(+,3)}_{\rm NLL} = i\pi \, \frac{(B_0)^3}{3!} \left[\frac{1}{(2\epsilon)^3} - \frac{11\zeta_3}{4} - \frac{33\zeta_4}{8}\epsilon - \frac{357\zeta_5}{4}\epsilon^2 + \mathcal{O}(\epsilon^3) \right] (C_A - \mathbf{T}^2_t)^2 \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(+,4)}_{\rm NLL} = i\pi \, \frac{(B_0)^4}{4!} \Big\{ (C_A - \mathbf{T}^2_t)^3 \left(\frac{1}{(2\epsilon)^4} + \frac{175\zeta_5}{2}\epsilon + \mathcal{O}(\epsilon^2) \right) + C_A (C_A - \mathbf{T}^2_t)^2 \left(-\frac{\zeta_3}{8\epsilon} - \frac{3}{16}\zeta_4 - \frac{167\zeta_5}{8}\epsilon + \mathcal{O}(\epsilon^2) \right) \Big\} \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)}.$$

Caron-Huot, 2013

- At four loop a new color structure appear, with a single pole not predicted by the dipole formula of infrared divergences!
- The fact that it arises only at four loops is a consequence of the "top-bottom" symmetry of the ladder. The new color structure appears in the target-averaged wave function already at three loops, but it cancels out due to this symmetry.



TWO REGGEON CUT: SOFT APPROXIMATION

- It would be possible to calculate few order higher in perturbation theory; the problem becomes rapidly quite involved.
- However, this is not necessary, if we are interested to know only the infrared singularities. Reconsider the wave function:

$$\Omega^{(\ell-1)}(p,k) = (2C_A - \mathbf{T}_t^2)\,\Psi^{(\ell-1)}(p,k) + (C_A - \mathbf{T}_t^2)\,\Phi^{(\ell-1)}(p,k),$$

with

$$\Psi^{(\ell-1)}(p,k) = \int [\mathrm{D}k']\, f(p,k,k') \left[\Omega^{(\ell-2)}(p,k') - \Omega^{(\ell-2)}(p,k)\right], \qquad \Phi^{(\ell-1)}(p,k) = \frac{1-J(p,k)}{2\epsilon}\Omega^{(\ell-2)}(p,k),$$

where

$$f(p,k',k) = \frac{k'^2}{k^2(k-k')^2} + \frac{(p-k')^2}{(p-k)^2(k-k')^2} - \frac{p^2}{k^2(p-k)^2},$$

$$J(p,k) = \left(\frac{p^2}{k^2}\right)^{\epsilon} + \left(\frac{p^2}{(p-k)^2}\right)^{\epsilon} - 1.$$

finite!

- The wave function is actually finite. All divergences must arise from the last integration!

$$\hat{\mathcal{M}}^{(+,\ell)}_{\mathrm{NLL}} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!} \int [\mathrm{D}k]\, \frac{p^2}{k^2(k-p)^2}\, \Omega^{(\ell-1)}(p,k)\, \mathbf{T}^2_{s-u}\, \mathcal{M}^{(0)},$$

- Divergences arises only from the limit $k \to p$ or $k \to 0$ limit. Consider one of the two regions, and multiply the result by two.

TWO REGGEON CUT: SOFT APPROXIMATION

- In the soft limit the integrations becomes trivial ("bubble" integrals). We obtain an all-order solution for the target-averaged wave function

$$\Omega_s^{(\ell-1)}(p,k) = \frac{(C_A - \mathbf{T}_t^2)^{\ell-1}}{(2\epsilon)^{\ell-1}} \sum_{n=0}^{\ell-1} (-1)^n \binom{\ell-1}{n} \left(\frac{p^2}{k^2}\right)^{n\epsilon} \prod_{m=0}^{n-1} \left\{1 + \hat{B}_m(\epsilon) \frac{2C_A - \mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}\right\},$$

where

$$\hat{B}_n(\epsilon) = \frac{B_n(\epsilon)}{B_0(\epsilon)} - 1, \quad \text{and} \quad B_n(\epsilon) = e^{\epsilon\gamma_E} \frac{\Gamma(1-\epsilon)}{\Gamma(1+n\epsilon)} \frac{\Gamma(1+\epsilon+n\epsilon)\Gamma(1-\epsilon-n\epsilon)}{\Gamma(1-2\epsilon-n\epsilon)}.$$

- It is immediate to get the reduced amplitude

$$\hat{\mathcal{M}}_{\mathrm{NLL}}^{(+,\ell)}|_s = i\pi \frac{1}{(2\epsilon)^\ell} \frac{B_0^\ell(\epsilon)}{\ell!} (1 + \hat{B}_{-1}) (C_A - \mathbf{T}_t^2)^{\ell-1} \sum_{n=1}^{\ell} (-1)^{n+1} \binom{\ell}{n}$$
$$\times \prod_{m=0}^{n-2} \left[1 + \hat{B}_m(\epsilon) \frac{2C_A - \mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}\right] \mathbf{T}_{s-u}^2 \, \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

- The result is valid up to the single poles, which allows one to achieve a tremendous simplification

$$\hat{\mathcal{M}}_{\mathrm{NLL}}^{(+,\ell)}|_s = i\pi \frac{1}{(2\epsilon)^\ell} \frac{B_0^\ell(\epsilon)}{\ell!} \left(1 - R(\epsilon) \frac{C_A}{C_A - \mathbf{T}_t^2}\right)^{-1} (C_A - \mathbf{T}_t^2)^{\ell-1} \, \mathbf{T}_{s-u}^2 \, \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

where

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$$R(\epsilon) \equiv \frac{B_0(\epsilon)}{B_{-1}(\epsilon)} - 1 = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\epsilon^3 - 3\zeta_4\epsilon^4 - 6\zeta_5\epsilon^5 - \left(2\zeta_3^2 + 10\zeta_6\right)\epsilon^6 + \mathcal{O}(\epsilon^7).$$

TWO REGGEON CUT: SOFT APPROXIMATION

- Expand for a few orders in the strong coupling constant:

$$\hat{\mathcal{M}}^{(+,\ell=1,2,3)}_{\rm NLL}|_s = i\pi \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} (C_A - \mathbf{T}_t^2)^{\ell-1} \mathbf{T}^2_{s-u}\,\mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

$$\hat{\mathcal{M}}^{(+,\ell=4,5,6)}_{\rm NLL}|_s = i\pi \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} \Big\{(C_A - \mathbf{T}_t^2)^{\ell-1} + R(\epsilon)\, C_A (C_A - \mathbf{T}_t^2)^{\ell-2}\Big\} \mathbf{T}^2_{s-u}\,\mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

$$\hat{\mathcal{M}}^{(+,\ell=7,8,9)}_{\rm NLL}|_s = i\pi \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} \Big\{(C_A - \mathbf{T}_t^2)^{\ell-1} + R(\epsilon)\, C_A (C_A - \mathbf{T}_t^2)^{\ell-2} + R^2(\epsilon)\, C_A^2 (C_A - \mathbf{T}_t^2)^{\ell-3}\Big\} \mathbf{T}^2_{s-u}\,\mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

$$\hat{\mathcal{M}}^{(+,\ell=10,11,12)}_{\rm NLL}|_s = i\pi \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} \Big\{(C_A - \mathbf{T}_t^2)^{\ell-1} + R(\epsilon)\, C_A (C_A - \mathbf{T}_t^2)^{\ell-2} + R^2(\epsilon)\, C_A^2 (C_A - \mathbf{T}_t^2)^{\ell-3} + R^3(\epsilon)\, C_A^3 (C_A - \mathbf{T}_t^2)^{\ell-4}\Big\} \mathbf{T}^2_{s-u}\,\mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

A new color structure appears every three loops!

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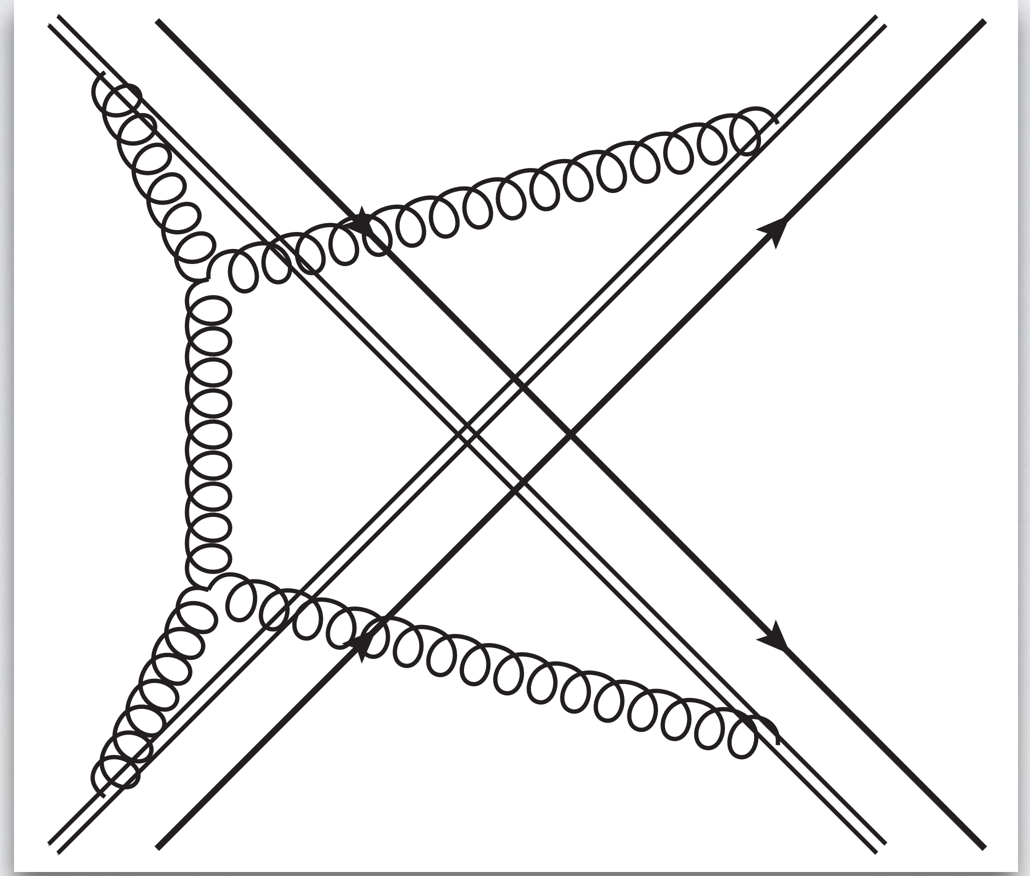
- Resumming the amplitude to all loops we get

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL}|_s = 4\pi\alpha_s \frac{i\pi}{L(C_A - \mathbf{T}_t^2)} \left(1 - R(\epsilon)\frac{C_A}{C_A - \mathbf{T}_t^2}\right)^{-1} \left[\exp\left\{\frac{B_0(\epsilon)}{2\epsilon}\frac{\alpha_s}{\pi} L (C_A - \mathbf{T}_t^2)\right\} - 1\right] \mathbf{T}^2_{s-u}\,\mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

COMPARISON BETWEEN REGGE AND INFRARED FACTORIZATION



TWO REGGEON CUT: BFKL VS INFRARED FACTORISATION

- Consider the soft anomalous dimension

$$\mathbf{\Gamma}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)=\tilde{\mathbf{\Gamma}}\left(\frac{s}{t},\lambda,\alpha_s(\lambda^2)\right)+\sum_{i=1}^{4}\mathbf{\Gamma}_i\left(t,\lambda,\alpha_s(\lambda^2)\right)+\mathcal{O}\left(\frac{t}{s}\right),$$

- with

$$\tilde{\mathbf{\Gamma}}\left(\alpha_s(\lambda^2)\right)=\tilde{\mathbf{\Gamma}}_{\rm LL}\left(\alpha_s(\lambda^2)\right)+\tilde{\mathbf{\Gamma}}_{\rm NLL}\left(\alpha_s(\lambda^2)\right)+\tilde{\mathbf{\Gamma}}_{\rm NNLL}\left(\alpha_s(\lambda^2)\right)+\ldots$$

- Parameterise the soft anomalous dimension at NLL according to

$$\tilde{\mathbf{\Gamma}}_{\rm NLL}\left(\alpha_s(\lambda^2)\right)=\sum_{\ell=1}^{\infty}\tilde{\mathbf{\Gamma}}^{(\ell)}_{\rm NLL}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^{\ell}=\sum_{\ell=1}^{\infty}\tilde{\mathbf{\Gamma}}^{(\ell)}_{\rm NLL}\left(\frac{\alpha_s(p^2)}{\pi}\right)^{\ell}\left(\frac{p^2}{\lambda^2}\right)^{\ell\epsilon}.$$

- Within the dipole formula one has

$$\tilde{\mathbf{\Gamma}}_{\rm LL}\left(\alpha_s(\lambda^2)\right)=\frac{\gamma_K\left(\alpha_s(\lambda^2)\right)}{2}\,L\,\mathbf{T}_t^2,\qquad\qquad \tilde{\mathbf{\Gamma}}^{(1)}_{\rm NLL}=i\pi\,\mathbf{T}^2_{s-u},$$

- Recall now the infrared factorisation formula

$$\mathcal{M}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)=\mathbf{Z}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

- with

$$\mathbf{Z}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)=\mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\mathbf{\Gamma}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\}.$$

TWO REGGEON CUT: BFKL VS INFRARED FACTORISATION

- We get the infrared-factorised representation of the reduced amplitude:

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL} = 4\pi\alpha_s \, \exp\left\{\frac{(B_0-1)}{2\epsilon}\frac{\alpha_s}{\pi}L(C_A-\mathbf{T}_t^2)\right\}\exp\left\{-\frac{1}{2\epsilon}\frac{\alpha_s}{\pi}L\mathbf{T}_t^2\right\}$$
$$\times\,\mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{p^2}\frac{{\rm d}\lambda^2}{\lambda^2}\left[\tilde{\mathbf{\Gamma}}_{\rm LL}\left(\alpha_s(\lambda^2)\right)+\tilde{\mathbf{\Gamma}}_{\rm NLL}\left(\alpha_s(\lambda^2)\right)\right]\right\}\mathcal{M}^{(0)}+\mathcal{O}(\epsilon^0),$$

- and comparing with the result from the Regge theory allows us to obtain

$$\tilde{\mathbf{\Gamma}}^{(\ell)}_{\rm NLL} = \frac{i\pi}{(\ell-1)!}\left[\frac{\alpha_s}{\pi}\left(1-R\left(\frac{\alpha_s}{2\pi}L(C_A-\mathbf{T}_t^2)\right)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1}\right]_{\alpha_s^\ell}\mathbf{T}^2_{s-u}.$$

- Explicitly, for the first few orders we have:

$$\tilde{\mathbf{\Gamma}}^{(1)}_{\rm NLL} = i\pi\,\mathbf{T}^2_{s-u}, \qquad \tilde{\mathbf{\Gamma}}^{(2)}_{\rm NLL} = 0, \qquad \tilde{\mathbf{\Gamma}}^{(3)}_{\rm NLL} = 0,$$
$$\tilde{\mathbf{\Gamma}}^{(4)}_{\rm NLL} = -i\pi\,L^3\,\frac{\zeta_3}{24}\,C_A(C_A-\mathbf{T}_t^2)^2\,\mathbf{T}^2_{s-u},$$
$$\tilde{\mathbf{\Gamma}}^{(5)}_{\rm NLL} = -i\pi\,L^4\,\frac{\zeta_4}{128}\,C_A(C_A-\mathbf{T}_t^2)^3\,\mathbf{T}^2_{s-u},$$
$$\tilde{\mathbf{\Gamma}}^{(6)}_{\rm NLL} = -i\pi\,L^5\,\frac{\zeta_5}{640}\,C_A(C_A-\mathbf{T}_t^2)^4\,\mathbf{T}^2_{s-u},$$
$$\tilde{\mathbf{\Gamma}}^{(7)}_{\rm NLL} = i\pi\frac{L^6}{720}\left[\frac{\zeta_3^2}{16}\,C_A^2(C_A-\mathbf{T}_t^2)^4+\frac{1}{32}\left(\zeta_3^2-5\zeta_6\right)\,C_A(C_A-\mathbf{T}_t^2)^5\right]\mathbf{T}^2_{s-u},$$
$$\tilde{\mathbf{\Gamma}}^{(8)}_{\rm NLL} = i\pi\frac{L^7}{5040}\left[\frac{3\zeta_3\zeta_4}{32}\,C_A^2(C_A-\mathbf{T}_t^2)^5+\frac{3}{64}\left(\zeta_3\zeta_4-3\zeta_7\right)\,C_A(C_A-\mathbf{T}_t^2)^6\right]\mathbf{T}^2_{s-u}.$$

Caron-Huot, Gardi, Reichel, LV, preliminar

Almelid, Duhr, Gardi, McLeod, White, 2017

- The result can be used as constraint in a bootstrap approach to the soft anomalous dimension.

CONCLUSION

- Using the non-linear Balitsky-JIMWLK rapidity evolution equation we have computed the three-Reggeon cut to three loops, at NNLL in the signature-odd sector, and the IR singular part of the two-Reggeon cut to all orders, at NLL in the signature-even sector, for $2 \to 2$ scattering amplitudes.
- Concerning the three-Reggeon cut, we have shown how to take systematically into account the effect of mixing between states with k and k+2 Reggeized gluons, due non-diagonal terms in the Balitsky-JIMWLK Hamiltonian, which contribute first at NNLL.
- Our results are consistent with a recent determination of the infrared structure of scattering amplitudes at three loops, as well as a computation of $2 \to 2$ gluon scattering in $N = 4$ super Yang-Mills theory. Combining the latter with our Regge-cut calculation we extract the three-loop Regge trajectory in this theory.
- The calculation of the infrared singular part of the two-Reggeon cut allows us to extract the soft anomalous dimension to all orders in perturbation theory, in this kinematical limit.
- The information obtained concerning infrared singularities has been/will be used to constrain the structure of the soft anomalous dimension in general kinematics. (See Almelid, Duhr, Gardi, McLeod, White, 2017).